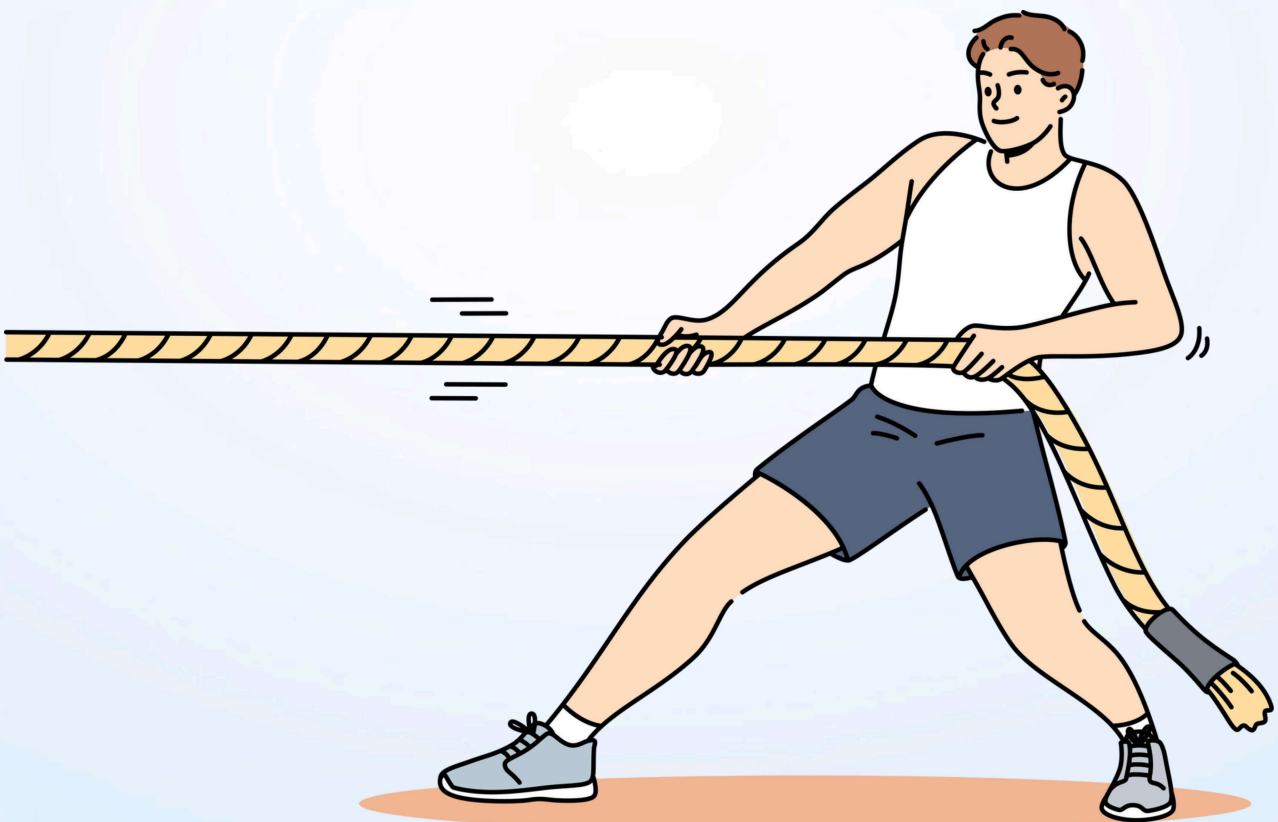


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ENGINEERING MECHANICS



BY JASPAL SIR

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ENGINEERING Mechanics

Study means!

It is the process of Observation, observation of either system, or object w.r to frame of Reference.

Rest - UN

It is the branch of physics that deals with the state of Rigid body under the application of external force.

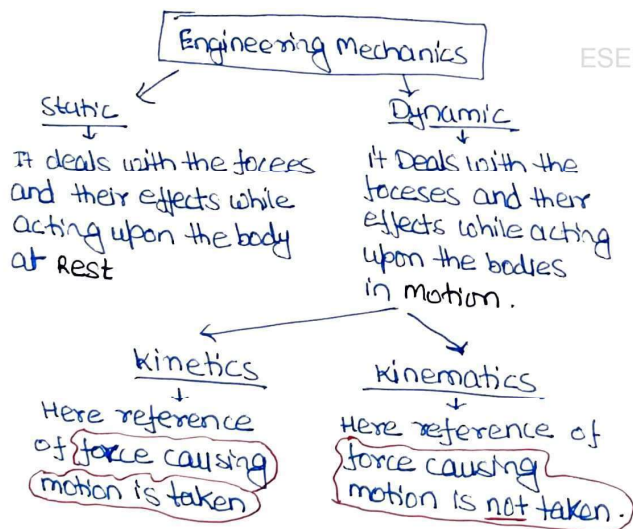
Here the condition of body is termed as State, which may be either! -

a) Rest

b) Motion

- Translational (Motion after applying brake in a BIKE)
- Rotation $\rightarrow$ (Motion when wheel rotate but no distance cover)
- General $\rightarrow$ (Translational + Rotation)

Here Rigid body signifies the ideal body which do not under go deformation over the application of force (External force)



In Order to define the STATE of body certain physical Parameters are required & termed as Physical Quantity, which are classified as! -

① Fundamental Physical Quantities :-

mass, Length, Time.

② Derived Physical Quantities! -

Area, Volume, Velocity, Acceleration, Force, Pressure, etc.

③ Scalar Physical Quantities! -

mass, Length, time, Distance, Density, Temperature, area, speed, Volume etc.

These are those Quantities which have only magnitude.

④ Vector Physical Quantities :-

Force, Displacement, velocity, Acceleration, momentum etc.

These are those quantities which have both magnitude and direction.

NOTE! -

A broader term includes both scalar and vector quantities is termed as Tensor.

These Tensor may be of following order / degree:-

- zero order / zero degree Tensor $\Rightarrow$ magnitude + zero direction
Scale
- First order / degree Tensor $\Rightarrow$ magnitude + 1-Direction
= Vector.
- Second Order / degree OR Tensor $\Rightarrow$ magnitude + 2-Direction
= Tensor
Example $\rightarrow$ Stress (σ_{xy}).



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UNITS:-

In order to measure the quantities an arbitrary internationally accepted parameter is required termed as UNIT.

These are classified as:-

(A) Fundamental Units:-

These are the units which are required for measurement of fundamental quantities. (mass, length, time)

NOTE:-

The above three fundamental quantities are dealt in engineering mechanics, but apart from them, electric current, Temperature, Luminous Intensity are also fundamental quantities.

(B) Derived Unit:-

These are the units which measure derived quantities

(Area, acceleration, velocity etc.)

System of Unit:-

In total four system of units are commonly used.

- CGS Unit (Centimeter, gram, seconds)
- MKS Unit (meter, kilogram, seconds)
- FPS Unit (Feet, Pound, seconds)
- SI Unit (International System of Unit)
(used now a day).

NOTE:-

SI Unit is same as MKS units system with slight variation in derived unit.

Example:-

Force = N instead of $\text{kg} \cdot \text{m}/\text{sec}^2$

work = N-m - Joule

Power = Watt - Joule/sec.

Now Fundamental unit are defined as follows:-

(i) metre:-

It is defined as shortest distance b/w two parallel line engraved upon the polished surface of Platinum-Radium Bar at 0°C . (kept at International Bureau of weight and measure headquarters in Paris)

(ii) Kilogram:-

It is defined as the mass of Platinum-Iridium Cylinder kept in Paris

(iii) Seconds:-

It is defined as $\left(\frac{1}{86400}\right)^{\text{th}}$ portion of mean solar day.

or $\left(\frac{1}{24 \times 60 \times 60}\right)^{\text{th}}$

→ at the time we see Sun, next day, at same time we see Sun, these time duration is called one solar day.

⇒ Avg of all 360 solar day's called mean solar day.



Vectors:-

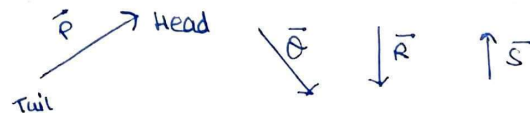
The vectors quantities are those quantities which have both magnitude and direction. Example - force, displacement etc.

It may also termed as First-order/degree rank tensor.

Representation of vector can be done by:-

(A) Graphical method:-

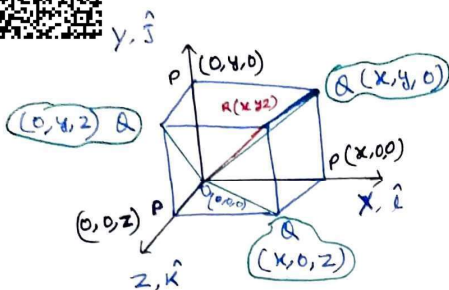
In this method vectors are represented by Arrows, when length of arrow signifies its magnitude and its orientation signifies its direction.



In order to carry out the complete analysis of vectors reference system is required, which are as follows:-

(1) Cartesian Co-ordinate system:-

P.T.O



In general:-

(a) For 1-Dimensional

$$\vec{P} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{Length} = \text{magnitude} = |\vec{P}| = |\vec{OP}|$$

$$= \sqrt{x^2 + y^2 + z^2}$$

(b) For 2-dimensional

$$\vec{P} = x\hat{i} + y\hat{j}; x\hat{i} + z\hat{k}; y\hat{j} + z\hat{k}$$

$$|\vec{P}| = |\vec{OP}| = \sqrt{x^2 + y^2} \text{ OR } \sqrt{x^2 + z^2} \text{ OR } \sqrt{y^2 + z^2}$$

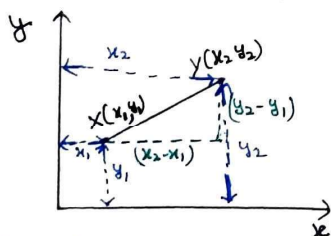
(c) In 3-dimensional

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{R}| = \sqrt{x^2 + y^2 + z^2}$$

NOTE:-

Distance b/w two point is given by (2D)



Acc. to Pythagoras:-

$$(xy)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$xy = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similarly in 3D

$$xy = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

UN-2

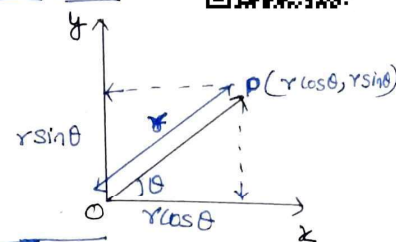
(2) Polar Co-ordinate system:-

(a) 2-D case

$$\vec{P} = \vec{OP} = x\hat{i} + y\hat{j}$$

$$\vec{OP} = r \cos \theta \hat{i} + r \sin \theta \hat{j}$$

$$|\vec{P}| = |\vec{OP}| = P = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} = r$$



NOTE

Here representation can also be done in terms of direction cosine, angle measure from

$$\vec{P} = \vec{OP} = x\hat{i} + y\hat{j} = r \cos \alpha \hat{i} + r \cos \beta \hat{j}$$

those axis, where you take a projection

$$|\vec{P}| = |\vec{OP}| = P = r = \sqrt{x^2 + y^2}$$

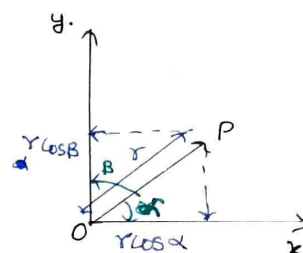
Now, $r \cos \alpha = x$

$$\cos \alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

Similarly

$$r \cos \beta = y$$

$$\cos \beta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$



Here, 'cos alpha' and 'cos beta' are termed as direction cosines.

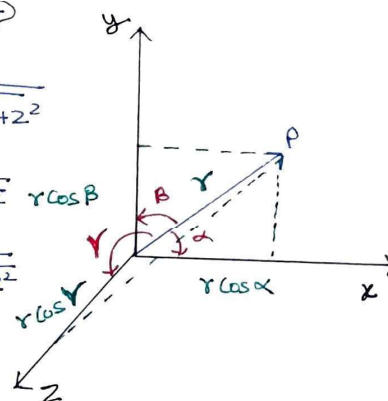
(b) In 3D case:-

$$\cos \alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \beta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \gamma = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Here cos alpha, cos beta, cos gamma are termed as direction cosine of the vector $\vec{OP}$



P.T.O

Now:-

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

also:- $\because \sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = 1 - \sin^2 \alpha$

$$1 - \sin^2 \alpha + 1 - \sin^2 \beta + 1 - \sin^2 \gamma = 1$$

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$$

$$\begin{aligned} \cos \alpha &= l \\ \cos \beta &= m \\ \cos \gamma &= n \end{aligned}$$

$$l^2 + m^2 + n^2 = 1$$

* if two lines are perpendicular then

$$\vec{a} \cdot \vec{b} = 0$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

⑤ Analytical method:-

$$\text{If } \vec{P} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{P}| = |\vec{OP}| = P = \sqrt{x^2 + y^2 + z^2}$$

$$\text{Now, } \vec{P} = P \cdot \hat{P}$$

vector $\leftarrow$ magnitude of vector P

Unit vector.

Hence, **unit vector** is the vector whose magnitude is unity and it signifies the direction of the vector.

$$\hat{P} = \frac{\vec{P}}{P} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

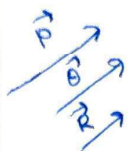
$$= \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{i} + \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{j} + \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \hat{k}$$

TYPES OF VECTOR

① Parallel Vector:-

means 'unit vector same'

Vector having same direction, but different magnitude (may be) are termed as parallel vector.



Here

$$\vec{P} = P \cdot \hat{P} = P \cdot \hat{U}$$

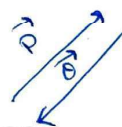
$$\vec{Q} = Q \cdot \hat{Q} = Q \cdot \hat{U}$$

$$\vec{R} = R \cdot \hat{R} = R \cdot \hat{U}$$

$$\hat{P} = \hat{Q} = \hat{R} = \hat{U} \text{ (unit vector)}$$

⑤ Anti Parallel Vector:-

Vector having opposite direction and different (or same) magnitude are termed as antiparallel vector.



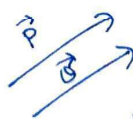
$$\vec{P} = P \cdot \hat{P}$$

$$\vec{Q} = Q \cdot \hat{Q}$$

$$\hat{P} = -\hat{Q}$$

⑥ Equal Vector:-

Vector having same direction and magnitude are termed as equal vector.



$$\vec{P} = P \cdot \hat{P}$$

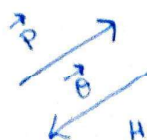
$$\hat{P} = \hat{Q}$$

$$P = Q$$

$$\text{Hence } \vec{P} = \vec{Q}$$

⑦ Opposite Vector:-

Vector having same magnitude but opposite direction are termed as opposite vector.



$$\vec{P} = P \cdot \hat{P}$$

$$\hat{P} = -\hat{Q}$$

$$P = Q$$

Hence,

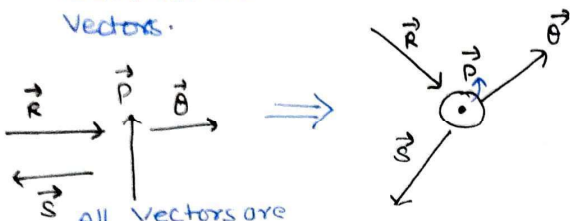
$$\vec{P} = -\vec{Q}$$

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e) Normal Vector :-

We can not comment on magnitude of these vector.

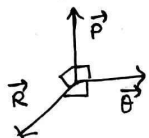
Vector which are perpendicular ($\perp$) to each other are termed as normal vectors.



All vectors are in different plane.

Note

If any 3-vectors are mutually $\perp$ to each other, they are termed as ORTHOGONAL VECTOR.



f) Null / zero Vector :-

A vector having zero magnitude is termed as null vector.

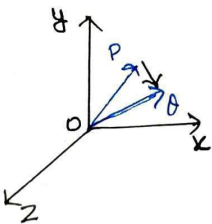
This vector has no specific direction.

g) Position Vector :-

A vector connection any position to the origin is termed as Position Vector.

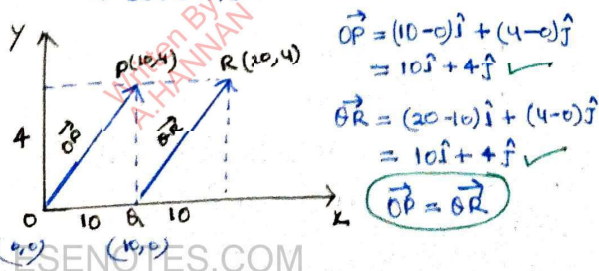
Here,

$\vec{OP}$, $\vec{OQ}$ are position vector but $\vec{PQ}$ is not position vector.



Note :- Translational Properties of Vector

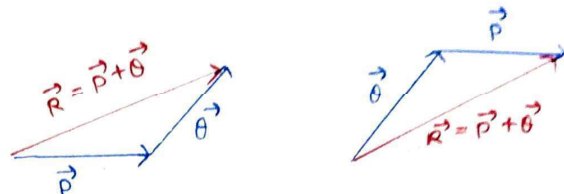
It is the property of the vector by the virtue of which it can be translated in space without changing its magnitude and direction.



ADDITION OF VECTOR

It can be done as follows :-

a) Graphical APPROACH :-



b) Analytical Approach :-

$$\vec{P} = p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$$

$$\vec{Q} = q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$

$$\vec{P} + \vec{Q} = (p_1 + q_1)\hat{i} + (p_2 + q_2)\hat{j} + (p_3 + q_3)\hat{k}$$

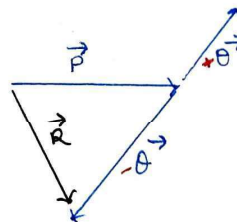
SUBTRACTION OF VECTOR

It can be achieved by addition of opposite vector.

$$\vec{R} = \vec{P} - \vec{Q} = \vec{P} + (-\vec{Q})$$

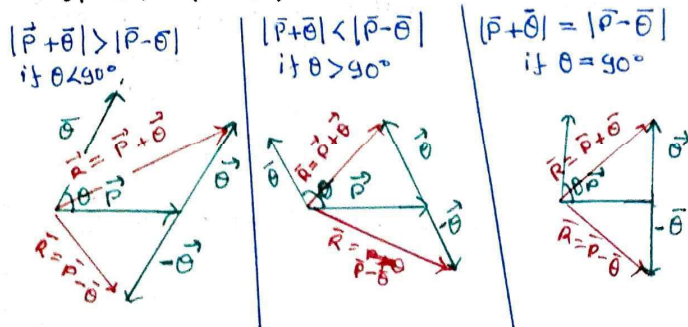
OR

$$\vec{R} = (p_1 - q_1)\hat{i} + (p_2 - q_2)\hat{j} + (p_3 - q_3)\hat{k}$$



NOTE :- In case of Vector :-

$$\left. \begin{aligned} |\vec{P} + \vec{Q}| &> |\vec{P} - \vec{Q}| \\ |\vec{P} + \vec{Q}| &< |\vec{P} - \vec{Q}| \\ |\vec{P} + \vec{Q}| &= |\vec{P} - \vec{Q}| \end{aligned} \right\} \text{all three cases are possible}$$



MULTIPLICATION OF VECTOR



a) Multiplication of vector with scalar:-

$$\vec{P} = p \cdot \hat{P}$$

i) Scalar with No Unit (k) Example $\rightarrow b, 2, b, a, b, etc.$

$$\vec{Q} = k \cdot \vec{P} = k \vec{P}$$

Here direction of $\vec{Q}$ is same as $\vec{P}$ and magnitude is "k" times magnitude of $\vec{P}$

ii) Scalar with Unit (k) Example $\rightarrow$ momentum $m\vec{v}$

$$\vec{Q} = k \cdot \vec{P} = k \vec{P}$$

$\vec{P} = m\vec{v}$
unit of $\vec{P} \neq$ unit of $\vec{Q}$

Here direction of $\vec{Q}$ is same as $\vec{P}$ and magnitude will be "k" times the magnitude of $\vec{P}$, unit but unit of $\vec{Q}$ will not be same as of $\vec{P}$.

UN-3

b) Multiplication of two vectors:-

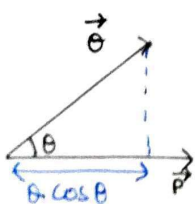
i) Scalar Product / Dot Product:-

It is the multiplication of two vectors results in scalar quantity. It is termed as Scalar Product.

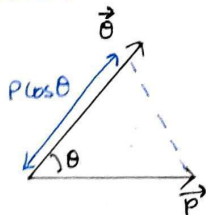
$$\vec{P} \cdot \vec{Q} = PQ \cos \theta$$

Here θ = angle b/w $\vec{P}$ and $\vec{Q}$

Dot Product signifies the multiplication of component of any one vector in the direction of other vector.



$$\vec{P} \cdot \vec{Q} = (Q \cos \theta) P$$



$$\vec{P} \cdot \vec{Q} = (P \cos \theta) Q$$

Example:-

$$\text{Work done (W)} = \text{Force (F)} \cdot \text{Displacement (S)}$$

(Scalar qty) (Vector qty) (Vector qty)

$$[\hat{i} \times \hat{j} = -\hat{k} ; \hat{k} \times \hat{j} = -\hat{i} ; \hat{j} \times \hat{k} = \hat{i}]$$

NOTE:-

$$a) \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

$$b) \vec{P} = p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k}$$

$$\vec{Q} = q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$$

$$\vec{P} \cdot \vec{Q} = p_1 q_1 + p_2 q_2 + p_3 q_3$$

$$c) \vec{P} \cdot \vec{Q} = PQ \cos \theta$$

Hence,

$$\cos \theta = \frac{\vec{P} \cdot \vec{Q}}{PQ} = \frac{p_1 q_1 + p_2 q_2 + p_3 q_3}{\sqrt{p_1^2 + p_2^2 + p_3^2} \sqrt{q_1^2 + q_2^2 + q_3^2}}$$

ii) Vector Product / Cross Product:-

If the multiplication of two vector results in vector quantity is termed as VECTOR/Cross product.

$$\vec{P} \times \vec{Q} = PQ \sin \theta \hat{u}$$

Magnitude $\rightarrow$ Direction

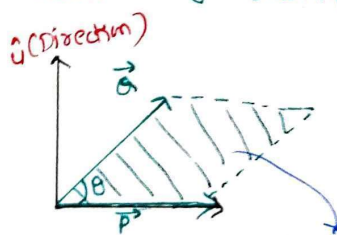
Here θ = angle b/w $\vec{P}$ and $\vec{Q}$

$\hat{u}$ = unit vector which $\perp$ to the plane $\vec{P}$ and $\vec{Q}$ and is found using Right Hand Thumb rule.

For Example:-

$$\text{Torque (}\vec{\tau}\text{)} = \text{Radius (}\vec{R}\text{)} \times \text{Force (}\vec{F}\text{)}$$

Vector Quantity Vector Quantity Vector Quantity



$$|\vec{P} \times \vec{Q}| = PQ \sin \theta = \text{area of parallelogram form by two vector.}$$

NOTE

$$a) \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k} ; \hat{j} \times \hat{k} = \hat{i} ; \hat{k} \times \hat{i} = \hat{j}$$

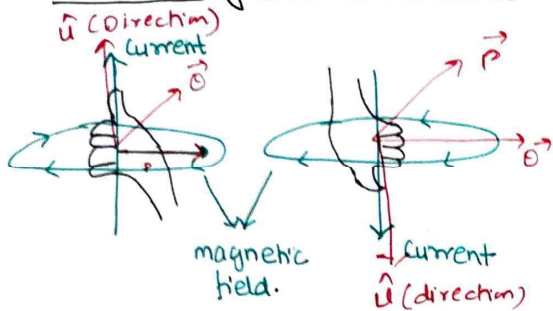
$$b) \vec{P} = p_1 \hat{i} + p_2 \hat{j} + p_3 \hat{k}$$

$$\vec{Q} = q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$$

$$\vec{P} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_2 q_3 - q_2 p_3) \hat{i} - (p_1 q_3 - q_1 p_3) \hat{j} + (p_1 q_2 - p_2 q_1) \hat{k}$$



Maxwell's Right Hand Thumb Rule :-



② Multiplication of Three Vector :-

Imp: ① Scalar Triple Product :-

For any three vector $\vec{P}, \vec{Q}, \vec{R}$ scalar triple product is defined as :-

$$(\vec{P} \times \vec{Q}) \cdot \vec{R} = \vec{P} \cdot (\vec{Q} \times \vec{R}) = \vec{Q} \cdot (\vec{R} \times \vec{P})$$

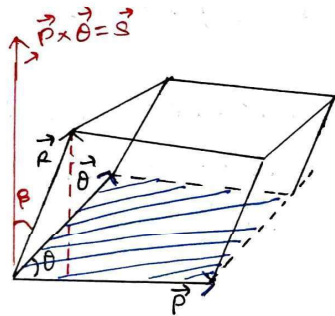
$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\vec{P} = p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$$

$$\vec{Q} = q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$

$$\vec{R} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$$

Since Product comes out to be scalar quantity it is termed scalar triple product.



$$(\vec{P} \times \vec{Q}) \cdot \vec{R} = \vec{S} \cdot \vec{R} = SR \cos \theta$$

' θ ' is angle between $\vec{S}$ and $\vec{R}$
Now $R \cos \theta$ is projection of $\vec{R}$ on $\vec{S}$
= height of Parallelepiped

Hence :-

$$(\vec{P} \times \vec{Q}) \cdot \vec{R} = (\text{Area of Parallelogram}) \times \text{Height}$$

= Volume of Parallelepiped formed by $\vec{P}, \vec{Q}$ and $\vec{R}$.

Hence :- If three vectors are coplanar, their scalar triple product is zero

② Vector Triple Product :-

If three vectors on multiplication results is vector quantity, it is termed as vector triple product.

It is given by

$$\vec{P} \times (\vec{Q} \times \vec{R}) = \vec{Q}(\vec{P} \cdot \vec{R}) - \vec{R}(\vec{P} \cdot \vec{Q})$$



Question ① :-

$$\text{If } \vec{P} = 6\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{Q} = 8\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\vec{S} = 4\hat{i} - 4\hat{j} + 4\hat{k}$$

Find the value of q_2 for with all three vectors will be coplanar.

For coplanar vector, scalar triple product is zero

$$\text{Hence } \vec{P} \cdot (\vec{Q} \times \vec{R}) = \begin{vmatrix} 6 & -2 & 2 \\ 8 & q_2 & -4 \\ 4 & -4 & 4 \end{vmatrix} = 0$$

$$6(4q_2 - 16) + 2(32 + 16) + 2(-32 - 4q_2) = 0$$

$$12q_2 - 48 + 32 + 16 - 32 - 4q_2 = 0$$

$$8q_2 = 32$$

$$q_2 = 4 \text{ Ans}$$

Question ② :-

$$\text{If } \vec{P} = 2\hat{i} - 4\hat{j} - 6\hat{k}$$

$$\vec{Q} = 2\hat{i} + 6\hat{j} + 4\hat{k}$$

$$\vec{S} = 4\hat{i} - 8\hat{j} + 2\hat{k}$$

compute :- a) Vector triple Product
b) scalar triple Product.

② Scalar Triple Product :-

$$\vec{P} \cdot (\vec{Q} \times \vec{S}) = \begin{vmatrix} 2 & -4 & -6 \\ 2 & 6 & 4 \\ 4 & -8 & 2 \end{vmatrix}$$

$$= 2(12 + 32) + 4(4 - 16) - 6(-16 - 24)$$

$$= 2 \times 44 + 4(-12) - 6(-40)$$

$$= 88 - 48 + 240$$

$$= 40 + 240 = 280$$

Q1) Vector triple product:-

$$\begin{aligned}\vec{P} \times (\vec{Q} \times \vec{R}) &= \vec{Q}(\vec{P} \cdot \vec{R}) - \vec{R}(\vec{P} \cdot \vec{Q}) \\ &= (2\hat{j} + 6\hat{j} + 4\hat{k})\{8 + 32 - 12\} - (4\hat{j} - 8\hat{j} + 2\hat{k})\{4 - 24 - 24\} \\ &= (2\hat{j} + 6\hat{j} + 4\hat{k})\{28\} - (4\hat{j} - 8\hat{j} + 2\hat{k})(-44) \\ &= (56\hat{j} + 168\hat{j} + 112\hat{k}) + (176\hat{j} - 352\hat{j} + 88\hat{k}) \\ &= [232\hat{j} - 184\hat{j} + 200\hat{k}] \text{ Ans}\end{aligned}$$

Q2) ③

If $\vec{P} = 4\hat{j} + 4\hat{j} + 4\hat{k}$ and $\vec{Q} = 2\hat{i} + 4\hat{j} + 2\hat{k}$
Find the angle between $\vec{P} + \vec{Q}$ and $\vec{P} - \vec{Q}$.

$$\begin{aligned}\vec{P} + \vec{Q} &= 6\hat{j} + 8\hat{j} + 6\hat{k} & |\vec{P} + \vec{Q}| &= \sqrt{6^2 + 8^2 + 6^2} \\ \vec{P} - \vec{Q} &= 2\hat{j} + 0\hat{j} + 2\hat{k} & &= \sqrt{36 + 64 + 36} \\ & & &= \sqrt{136} \\ & & &= 11.66 \\ \cos \theta &= \frac{(\vec{P} + \vec{Q}) \cdot (\vec{P} - \vec{Q})}{|\vec{P} + \vec{Q}| |\vec{P} - \vec{Q}|} & |\vec{P} - \vec{Q}| &= \sqrt{2^2 + 2^2} \\ &= \frac{(12 + 12)}{11.66 \times 2.83} & &= \sqrt{8} \\ &= 0.73 & &= 2.83 \\ \theta &= \cos^{-1}(0.73) = 43.3^\circ \text{ Ans}\end{aligned}$$

Q3) ④ If $\vec{P} = 4\hat{i} + 2\hat{j} + 4\hat{k}$
 $\vec{Q} = 2\hat{i} - 6\hat{j} + 2\hat{k}$,

then find the projection of $\vec{Q}$ in the direction of $\vec{P} + \vec{Q}$.

Projection of $\vec{Q}$ means $Q \cos \theta$ ($\because \vec{A} \cdot \vec{B} = AB \cos \theta$)

$$\begin{aligned}(\vec{P} + \vec{Q}) \cdot \vec{Q} &= (P + Q) \cos \theta \\ Q \cos \theta &= \frac{(\vec{P} + \vec{Q}) \cdot \vec{Q}}{(P + Q)} \\ &= \frac{(6\hat{i} - 4\hat{j} + 6\hat{k})(2\hat{i} - 6\hat{j} + 2\hat{k})}{\sqrt{6^2 + (-4)^2 + 6^2}}\end{aligned}$$

$$\cos \theta = \frac{12 + 24 + 12}{\sqrt{36 + 16 + 36}} = \frac{48}{\sqrt{88}}$$

$$\theta \cos \theta = \frac{48}{\sqrt{88}} = 5.11 \text{ Ans}$$

Q4) ⑤

Determine unit vector parallel to the sum of $\vec{P}$ and $\vec{Q}$ vector.

where -

$$\vec{P} = 4\hat{j} + 16\hat{j} - 10\hat{k}$$

$$\vec{Q} = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

$$\vec{P} + \vec{Q} = 6\hat{j} + 20\hat{j} - 4\hat{k}$$

$$\therefore \vec{P} = P \cdot \hat{P}$$

$$(\vec{P} + \vec{Q}) = (P + Q) (\hat{P} + \hat{Q})$$

$$(\hat{P} + \hat{Q}) = \frac{\vec{P} + \vec{Q}}{(P + Q)}$$



$$= \frac{6\hat{j} + 20\hat{j} - 4\hat{k}}{\sqrt{6^2 + 20^2 + (-4)^2}}$$

$$= \frac{6\hat{j} + 20\hat{j} - 4\hat{k}}{\sqrt{36 + 400 + 16}}$$

$$(\hat{P} + \hat{Q}) = \frac{6\hat{j} + 20\hat{j} - 4\hat{k}}{\sqrt{452}}$$

$$(\hat{P} + \hat{Q}) = \frac{6\hat{j}}{21.26} + \frac{20\hat{j}}{21.26} + \frac{(-4\hat{k})}{21.26} \text{ Ans}$$

Properties of scalar triple product:-

- $[\vec{a} \vec{b} \vec{c}] = [\vec{b} \vec{c} \vec{a}] = [\vec{c} \vec{a} \vec{b}]$ in cyclic order
- $[\vec{a} \vec{b} \vec{c}] = -[\vec{a} \vec{c} \vec{b}] = -[\vec{b} \vec{a} \vec{c}]$ - Non-cyclic order
- If coplanar vectors, then $[\vec{a} \vec{b} \vec{c}] = 0$
- If two vectors are same then also $[\vec{a} \vec{b} \vec{c}] = 0$

FORCES

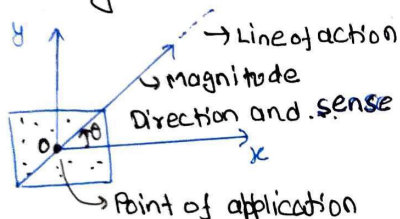


It may be defined as an agent which produces or tends to produce, destroys or tends to destroy the motion.

The force may produce the following effects in a body:-

- It may retard the motion of body.
- It may bring the body in motion from rest.
- It may accelerate the body which is already in motion.
- It may also give rise to internal stresses in the body on which it act.
- It may change the direction, magnitude or both of the force already acting on it.

In order to determine the effect of force acting on the body, we must know the following characteristics of force.

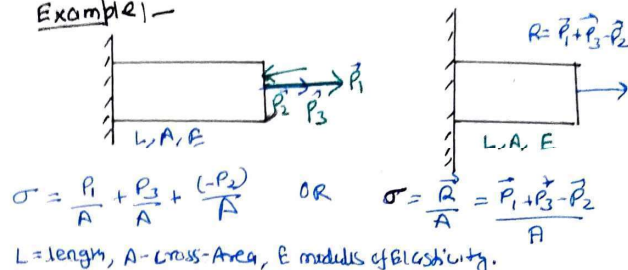


- Magnitude:-
It is measure of length of vector.
- Point of application:-
It is the point on body at which force is applied.
- Direction:-
It is measure angle made by force with x-axis.
- Sense:-
It is the measure of direction with it sign i.e. anticlockwise or clockwise.
- Line of Action:-
It is the line along the length of force vector.

Principle of Physical Independence of forces:-

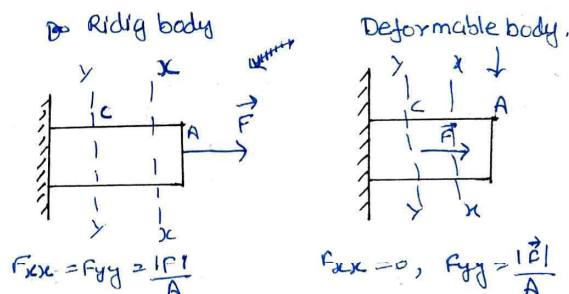
It states that if a number of forces are simultaneously acting on a body, then the resultant of these forces will have same effects as produced by all the forces.

Example 1:-



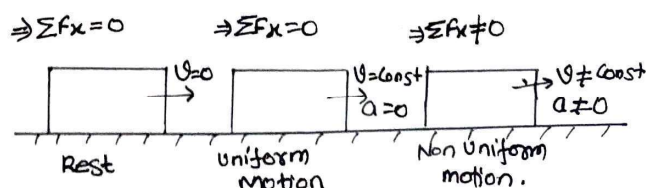
Principle of Transmissibility of forces

It state, that if any force is acting on a rigid body at any point, it can be transferred on other points on the same body without changing its magnitude, direction and line of action in order to get the same effect.



Newton's Law of motion:-

- 1st Law:- If body is in rest or uniform motion, it will remain in same state, until a net external force is applied on it.



[ΣF_x = External Forces.]

③ 2nd Law! — The applied force on the body is given by the rate of change of linear momentum of the body.

Hence —

(Linear momentum is given by $\vec{p} = m\vec{v}$)

$$\vec{F} = \frac{d(\vec{p})}{dt} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} + \vec{v} \frac{dm}{dt}$$

Case(I) :-

If mass is constant ($m=c$) $\Rightarrow \frac{dm}{dt} = 0$

$$\vec{F} = m \frac{d\vec{v}}{dt} = m\vec{a}$$



$$F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = m(a_x \hat{i} + a_y \hat{j} + a_z \hat{k})$$

Here —

$$F_x = ma_x$$

$$F_y = ma_y$$

$$F_z = ma_z$$

Case(II)

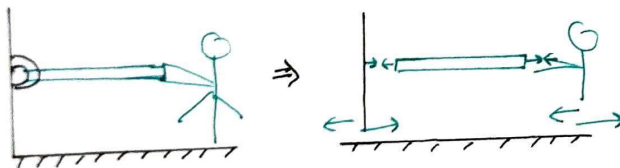
If velocity is constant ($v=c$) $\Rightarrow \frac{dv}{dt} = 0$

$$\vec{F} = \vec{v} \frac{dm}{dt}$$

Here $\frac{dm}{dt}$ represent mass flow rate (m)

Case(III)

③ 3rd Law! — Each action has equal and opposite reaction (but it will be acting on different bodies)



Written By
A.HANNAN

System of Forces! —

When two or more forces act on a body, they are called to form a system of forces.

Following are different system of forces: —

A Coplanar System Of force (2D): —

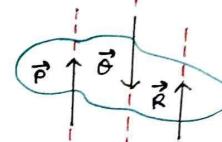
Coplanar forces: —

The forces, whose line of action lies on the same plane are termed as coplanar forces.

It also consist of following System

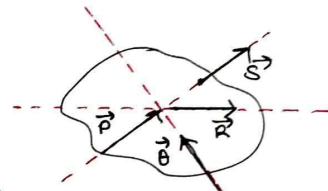
(i) Coplanar Parallel System: —

In this system line of action of forces lies on same plane and are parallel to each other.



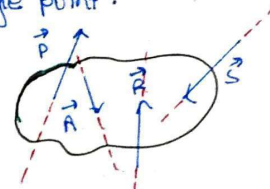
(ii) Coplanar Concurrent System: —

In this system line of action of force lies in same plane and meet each other at same point.



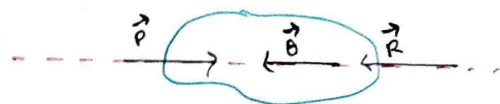
(iii) Coplanar Non-parallel non-concurrent system: —

In this system line of action of forces lies in this same plane but are neither parallel to each other nor meet's at a single point.



(iv) Coplanar Collinear System: —

In this system line of action of forces lies in the same plane and also on same line.



B Non-Coplanar System of force (3D):

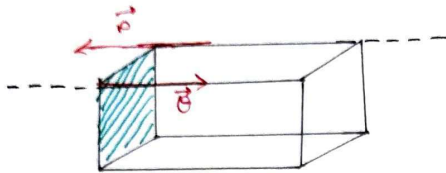
In this system line of action of forces will not be in same plane.

It further Includes:-



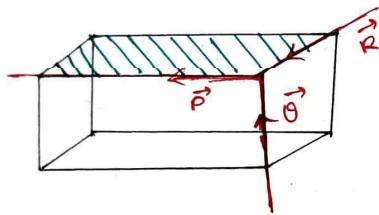
(i) Non coplanar Parallel System:-

In this system line of action of force will not be in same plane but would be parallel to each other.



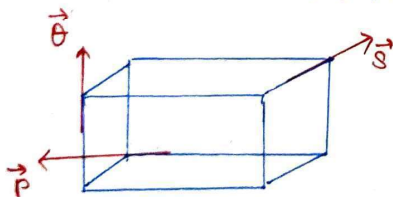
(ii) Non coplanar Concurrent System:-

In this system line of action of different force will not be in same plane but would meet each other



(iii) Non coplanar non parallel non concurrent system:-

In this system line of action of different force neither lie in same plane nor are parallel to each other nor meet at a point.



Written By
A.HANNAN

Resultant Force:-

If number of forces are acting simultaneously on a particle then it is possible to find out a single force which could replace all these force. i.e. it would produce the same effect as produced by different forces.

Here this single force is termed as resultant forces the given forces are termed as component force

The process of finding out the resultant force of number of given forces is called composition / Compounding of forces.

The resultant of forces can be found by following methods:-

- Analytical mtd.
- Graphical mtd.

A) Analytical method:-

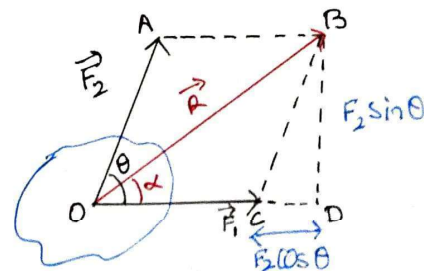
Resultant of forces can be found by analytical method as follows

- Parallelogram Law of Forces.
- Method of Resolution.

UN-5

a) Parallelogram Law of forces:-

It states, If two forces acting simultaneously on a particle be resultant in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represent in magnitude and direction by the diagonal of the parallelogram which passes through their point of intersection



P.T.O

By Jaspal Sir

In $\triangle OBD$,

$$OB^2 = BD^2 + OD^2$$

$$R^2 = (F_2 \sin \theta)^2 + (F_1 + F_2 \cos \theta)^2$$

$$R^2 = F_2^2 \sin^2 \theta + F_1^2 + F_2^2 \cos^2 \theta + 2F_1 F_2 \cos \theta$$

$$R^2 = F_1^2 + F_2^2 (\sin^2 \theta + \cos^2 \theta) + 2F_1 F_2 \cos \theta$$

$$R^2 = F_1^2 + F_2^2 + 2F_1 F_2 \cos \theta$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos \theta}$$



$$\tan \alpha = \frac{BD}{OD} = \frac{F_2 \sin \theta}{F_1 + F_2 \cos \theta}$$

$$\alpha = \tan^{-1} \frac{F_2 \sin \theta}{(F_1 + F_2 \cos \theta)}$$

"F₁ is always bigger force" α touch the force is here

NOTE If the angle (α) which resultant force makes with the other force F_2 then $\alpha = \tan^{-1} \frac{F_1 \sin \theta}{F_2 + F_1 \cos \theta}$

$$\text{if } \theta = 0 \Rightarrow R = F_1 + F_2$$

$$\theta = 90^\circ \Rightarrow \sqrt{F_1^2 + F_2^2} = R$$

$$\theta = 180^\circ \Rightarrow R = F_1 - F_2$$

$$F_1 = F_2 \Rightarrow R = \sqrt{F^2 + F^2 + 2F^2 \cos \theta}$$

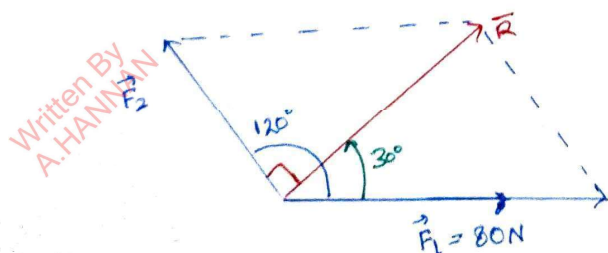
$$R = \sqrt{2F^2 + 2F^2 \cos \theta}$$

$$= \sqrt{2F^2 (1 + \cos \theta)}$$

$$= \sqrt{2F^2 \cdot 2 \cos^2 \theta/2}$$

$$R = 2F \cos \theta/2$$

Question:- Two forces act on an angle of 120° . The bigger force is of 80N and the resultant is perpendicular to the smaller one. Find the smaller force.



$$\alpha = \tan^{-1} \frac{F_2 \sin \theta}{F_1 + F_2 \cos \theta}$$

$$F_1 + F_2 \cos \theta$$

$$\tan 30 = \frac{F_2 \sin 120}{80 + F_2 \cos 120} = \frac{F_2 \sin (90+30)}{80 + F_2 \cos (90+30)}$$

$$\frac{1}{\sqrt{3}} = \frac{F_2 \cdot \frac{\sqrt{3}}{2}}{80 - F_2 \cdot \frac{1}{2}}$$

$$F_2 = 160 \text{ N} \Rightarrow \frac{1}{\sqrt{3}} = \frac{F_2 \cdot \frac{\sqrt{3}}{2}}{160 - \frac{F_2}{2}}$$

$$\Rightarrow 160 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} F_2$$

$$\Rightarrow F_2 = 160 \text{ N} = 40 \text{ Ans}$$

Ques 2 Compute the magnitude of two forces, such that if they act at right angles, their resultant is $\sqrt{20}\text{N}$. But if they act at 60° their resultant is $\sqrt{29}\text{N}$.

if $\theta = 90^\circ$

$$R = \sqrt{F_1^2 + F_2^2}$$

$$\sqrt{20} = \sqrt{F_1^2 + F_2^2}$$

$$F_1^2 + F_2^2 = 20 \quad \text{--- (i)}$$

if $\theta = 60^\circ$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos \theta}$$

$$\sqrt{29} = \sqrt{F_1^2 + F_2^2 + 2F_1 F_2 \cos 60}$$

$$29 = F_1^2 + F_2^2 + 2F_1 F_2 \cdot \frac{1}{2}$$

$$F_1^2 + F_2^2 + F_1 F_2 = 29 \quad \text{--- (ii)}$$

$$\therefore (F_1 + F_2)^2 = (F_1^2 + F_2^2) + 2F_1 F_2$$

$$= 20 + 29 = 49$$

from (i) and (ii)

$$20 + F_1 F_2 = 29$$

$$F_1 F_2 = 29 - 20 = 9$$

$$\therefore (F_1 + F_2)^2 = (F_1^2 + F_2^2) + 2F_1 F_2$$

$$(F_1 + F_2)^2 = 20 + 2 \times 9$$

$$(F_1 + F_2)^2 = 38 \text{ N} \quad \text{--- (A)}$$

$$(F_1 - F_2)^2 = (F_1^2 + F_2^2) - 2F_1 F_2$$

$$= 20 - 2 \times 9$$

$$(F_1 - F_2)^2 = 2 \quad \text{--- (B)}$$

From (A) and (B)

$$F_1 = 3.78 \text{ N}$$

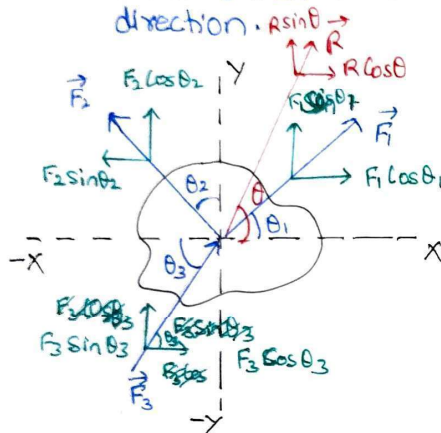
$$F_2 = 2.38 \text{ N} \quad \text{Ans}$$

⑥ Resolution of Forces:—

The process of splitting of the force into number of components, without changing its effect on the body is termed as Resolution of Force.

A force is generally resolved in two perpendicular direction.

Here Principle of Resolution: state that the algebraic sum of the resolved parts of number of forces in a given direction is equal to the resolved parts of their resultant in the same direction.



Resultant force of x-axis —

$$F_1 \cos \theta_1 + F_3 \sin \theta_3 - F_2 \sin \theta_2 = R \cos \theta$$

Resultant force of y-axis —

$$F_1 \sin \theta_1 + F_2 \cos \theta_2 + F_3 \cos \theta_3 = R \sin \theta$$

$$F_1 \sin \theta_1 + F_2 \cos \theta_2 + F_3 \sin \theta_3 = R \sin \theta$$

OR

$$R_x = \sum F_x \quad R_y = \sum F_y$$

$$R = \sqrt{\sum F_x^2 + \sum F_y^2}$$

$$\theta = \tan^{-1} \left(\frac{\sum F_y}{\sum F_x} \right)$$

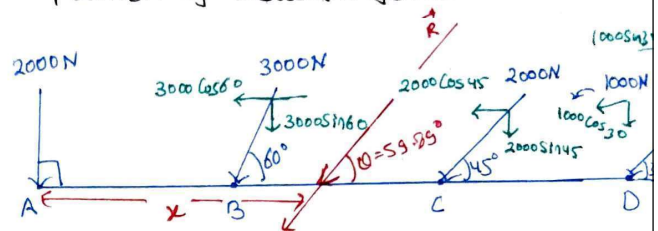
* If $\sum F_y$ is (+ve) ; $\theta \Rightarrow [0^\circ - 180^\circ]$
 $\sum F_y$ is (-ve) ; $\theta \Rightarrow [180^\circ - 360^\circ]$

* If $\sum F_x$ is (+ve) ; $\theta \Rightarrow [0^\circ - 90^\circ]$ and $[270^\circ - 360^\circ]$
 $\sum F_x$ is (-ve) ; $\theta \Rightarrow [90^\circ - 270^\circ]$

Question (3) Imp

A Horizontal plane ABCD is 12m. long where $AB = BC = CD = 4m$. Forces of 2000N, 3000N, 2000N and 1000N act at A, B, C, D respectively with downward direction. The line of action of these forces makes an angle $90^\circ, 60^\circ, 45^\circ, 30^\circ$ respectively with AD.

Compute the magnitude, direction and position of resultant force.



$$\begin{aligned} \sum F_x &= (3000 \cos 60^\circ + 2000 \cos 45^\circ + 1000 \cos 30^\circ) \\ &= (3000 \times \frac{1}{2} + 2000 \times \frac{\sqrt{2}}{2} + 1000 \times \frac{\sqrt{3}}{2}) \\ &= -3.78 \text{ kN} \end{aligned}$$

$$\begin{aligned} \sum F_y &= (2000 + 3000 \sin 60^\circ + 2000 \sin 45^\circ + 1000 \sin 30^\circ) \\ &= (2000 + 3000 \times \frac{\sqrt{3}}{2} + 2000 \times \frac{\sqrt{2}}{2} + 1000 \times \frac{1}{2}) \\ &= -6.52 \text{ kN} \end{aligned}$$

$$\begin{aligned} \therefore \sum F_x &= (-) \Rightarrow (90^\circ - 270^\circ) \\ \sum F_y &= (-) \Rightarrow (180^\circ - 360^\circ) \end{aligned} \quad \left. \begin{array}{l} \text{Common} \\ \downarrow \\ (180^\circ - 270^\circ) \end{array} \right\} \downarrow$$

Hence Resultant lies between $(180^\circ - 270^\circ)$

3rd Quadrant

$$\begin{aligned} \text{Resultant} &= \sqrt{(3.78)^2 + (6.52)^2} \\ &= 7.52 \text{ kN. (Magnitude)} \end{aligned}$$

$$\theta = \tan^{-1} \left(\frac{6.52}{3.78} \right) = 59.89^\circ$$

Since, $\sum F_x$ and $\sum F_y$ are (-ve)

$$\Rightarrow \theta = 180^\circ + 59.89^\circ$$

$$= 239.89^\circ \text{ (direction)}$$

Position of Resultant force (x) from A

Let point of application be at distance 'x' from A.

$$R_y \cdot x = (2 \times 0) + (3 \sin 60^\circ \times 4) + (2 \sin 45^\circ \times 8) + (1 \sin 30^\circ \times 12)$$

$$7.52 \cdot \sin(59.89^\circ) \cdot x = 28.57$$

$$x = \frac{28.57}{6.505} = 4.39 \text{ m}$$

Ans



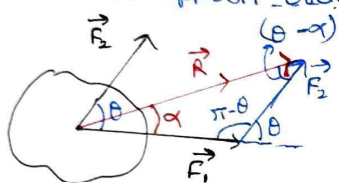
(B) Graphical method:-

Laws for resultant force.

- 1) Triangle Law of Forces
- 2) Polygon Law of Forces

① Triangle Law of Forces:-

It states, if two forces acting simultaneously on a particle, be represented in magnitude and direction by two side of triangle taken in order (direction) (sense i.e. clockwise or anticlockwise), then their resultant may be represented in magnitude and direction by the 3rd side of triangle, taken in opposite order (direction).

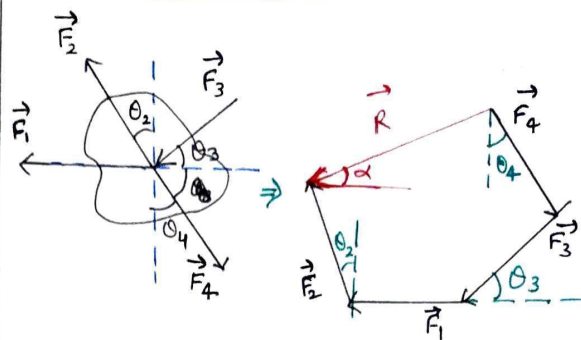


$$\left[\frac{F_1}{\sin(\theta - \alpha)} = \frac{F_2}{\sin \alpha} = \frac{R}{\sin(\pi - \theta)} \right] \text{ From these we can find 'R' and 'alpha'}$$

② Polygon Law of Forces:-

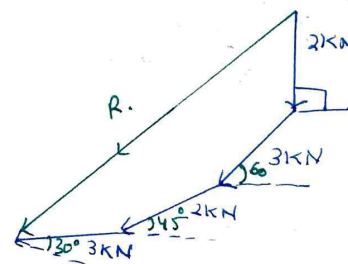
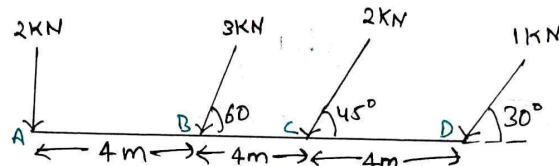
It is an extension of triangle law of forces.

It states, if more than two forces (number of forces) acting simultaneously be represented in magnitude and direction by the side of the polygon taken in an order, then the resultant of all these forces may be represented in magnitude and direction by closing side of polygon in opposite order.

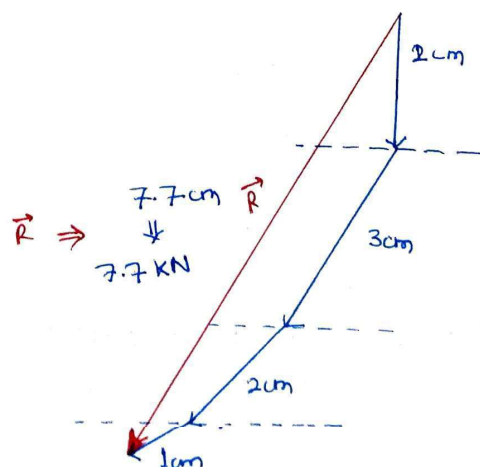


Graphical method to find resultant of forces is same as polygon law of forces.

Ques:- Solve the last previous problem by using graphical method.



Scale
1 cm = 1 kN



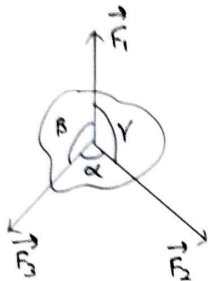
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NOTE 1-

Lami's Theorem:-



a) It states, if three coplanar, concurrent, non-linear, radial forces acts on a body and keep it in equilibrium then the forces will be proportional to the sine of their opposite angles.



$$F_1 \propto \sin \alpha; F_2 \propto \sin \beta, F_3 \propto \sin \gamma.$$

$$F_1 = K \sin \alpha$$

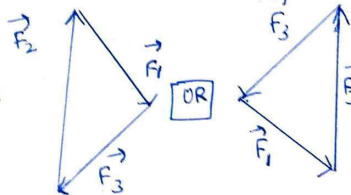
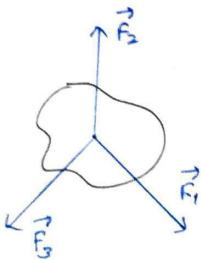
$$F_2 = K \sin \beta$$

$$F_3 = K \sin \gamma$$

$$K = \frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$$

Here $\alpha, \beta, \gamma \neq 180^\circ$

b) If ~~three~~ coplanar, concurrent, non-linear radial forces acts on a body in equilibrium, then the forces will form a closed triangle taken into order.



order $\curvearrowright$ order $\curvearrowleft$

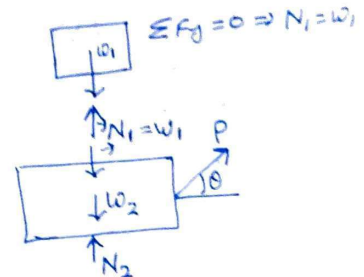
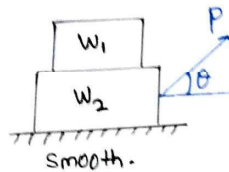
Hence, Here Resultant force = 0

Free body diagram:-

It is the diagram which obtained when a body is separated from all contacts and all forces (active and Reactive) are applied on.

which tries to move the body

which oppose the motion of body.



$$\sum F_y = 0 \Rightarrow N_1 = W_1$$

$$\sum F_y = 0 \Rightarrow W_1 + W_2 - N_2 - P \sin \theta = 0$$

$$N_2 = W_1 + W_2 - P \sin \theta //$$

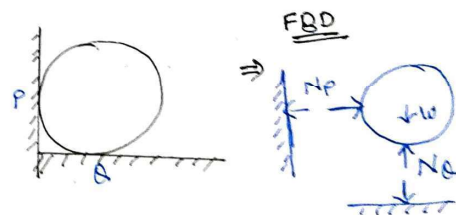
$$\sum F_x = 0 \Rightarrow P \cos \theta = 0 //$$

*Normal force (N_1, N_2) are always try to separate the body

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Ques 1) Compute the normal Reaction at point 'P' and 'Q'. where sphere of 'W' weight is in contact with the smooth surfaces.

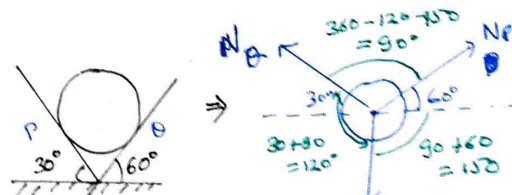
(a)



$$\sum F_x = 0 \Rightarrow N_p = 0$$

$$\sum F_y = 0 \Rightarrow N_q = W$$

(b)



1st method

$$\sum F_x = 0$$

$$\sum F_y = 0$$

2nd Lami's Law

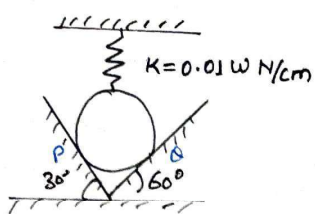
$$\frac{W}{\sin 90} = \frac{N_q}{\sin 150} = \frac{N_p}{\sin 120}$$

$$\frac{W}{1} = \frac{N_q \times 2}{\sqrt{3}} = \frac{N_p \times 2}{1}$$

$$N_q = \frac{\sqrt{3}W}{2}; N_p = \frac{W}{2}$$

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(C)



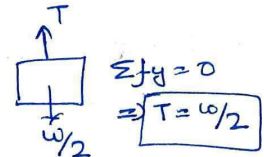
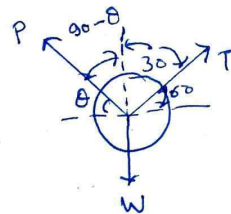
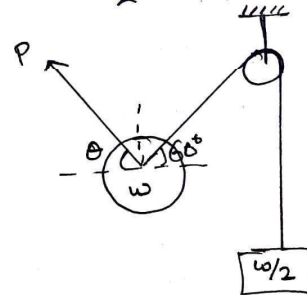
compression in spring is 75 cm.



Lami's theorem:-

$$\frac{W}{\sin 120} = \frac{T}{\sin 120} = \frac{N_0}{\sin 120}$$

Ques 3 Find the force and its direction for the equilibrium of system



$$\sum f_y = 0 \\ \Rightarrow T = W/2$$

Lami's theorem:-

$$\frac{W/2}{\sin(90+\theta)} = \frac{W}{\sin(90+30-\theta)} = \frac{P}{\sin(150)}$$

$$\frac{W/2}{\cos \theta} = \frac{W}{\sin(120-\theta)} = \frac{P}{\sin(150)}$$

$$\frac{W/2}{\cos \theta} = \frac{W}{\sin(120-\theta)} = \frac{P}{\sin 150}$$

$$\frac{W/2}{\cos \theta} = \frac{W}{\sin(90+(30-\theta))} = \frac{P}{\sin(30+\theta)}$$

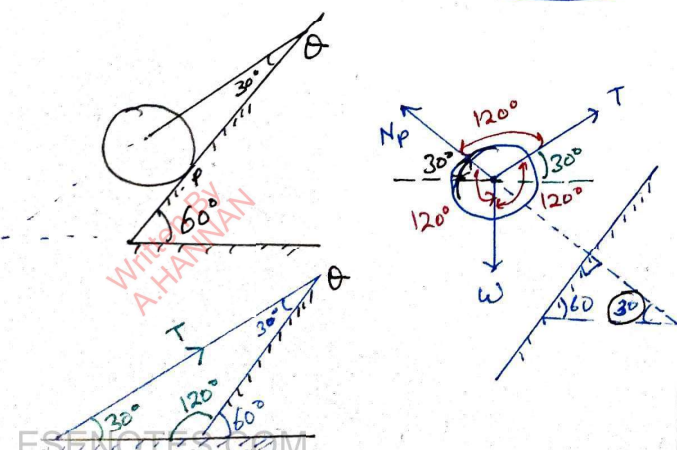
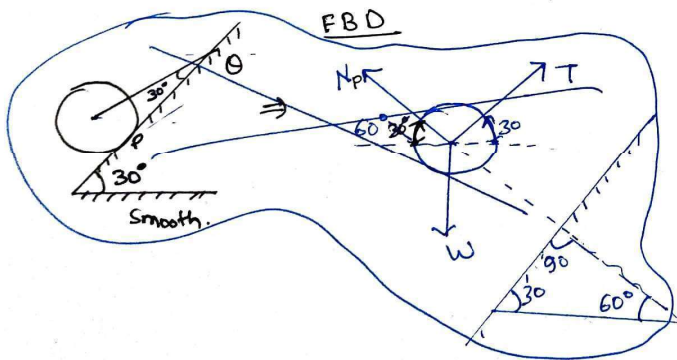
$$\frac{W/2}{\cos \theta} = \frac{7W}{\cos \theta}$$

Lami's theorem

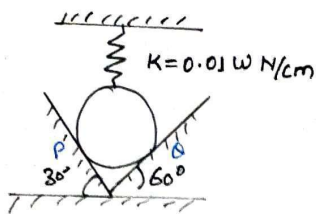
$$\frac{1.75W}{\sin 90} = \frac{N_p}{\sin 120} = \frac{N_0}{\sin 150}$$

$$N_p = ? \\ N_0 = ?$$

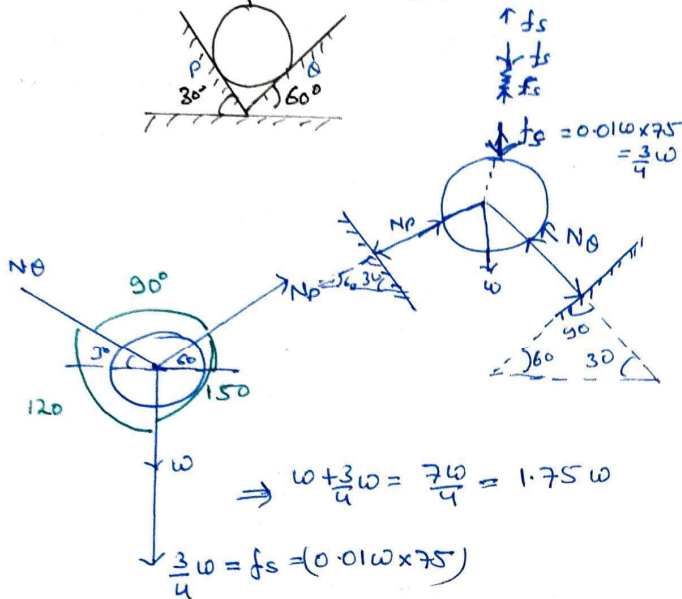
Ques 2 Compute the tension in the spring and also find the normal reaction b/w plane and sphere.



(C)



compression in spring is 7.5 cm.

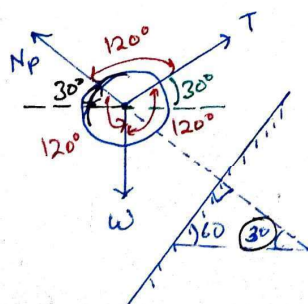
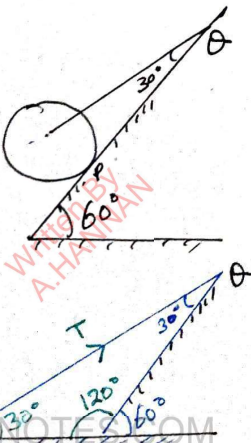
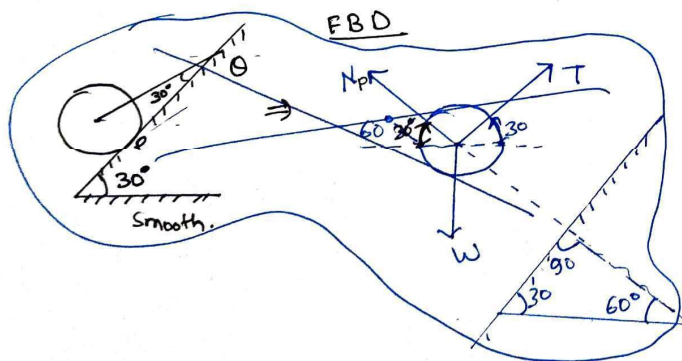


Lami's theorem

$$\frac{1.75W}{\sin 90} = \frac{N_p}{\sin 120} = \frac{N_g}{\sin 150}$$

$N_p = ?$
 $N_g = ?$

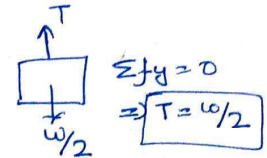
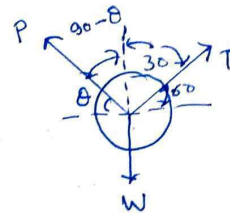
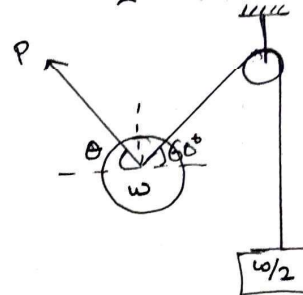
Que (2) Compute the tension in the spring and also find the normal reaction b/w plane and sphere.



Lami's theorem

$$\frac{W}{\sin 120} = \frac{T}{\sin 120} = \frac{N_g}{\sin 120}$$

Que (3) Find the force and its direction for the equilibrium of system



Lami's theorem

$$\frac{W/2}{\sin(90+\theta)} = \frac{W}{\sin(90+30+\theta)} = \frac{P}{\sin(150)}$$

$$\frac{W/2}{\sin(90+\theta)} = \frac{W}{\sin(90+30+\theta)} = \frac{P}{\sin(150)}$$

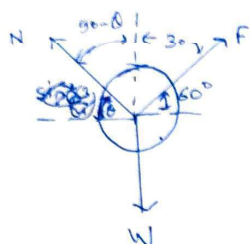
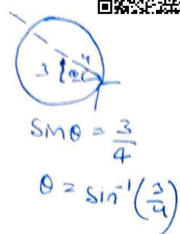
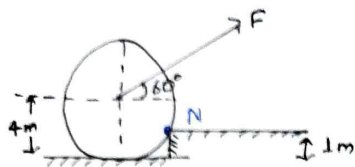
$$\frac{W/2}{\cos \theta} = \frac{W}{\sin(120+\theta)} = \frac{P}{\sin 150}$$

$$\frac{W/2}{\cos \theta} = \frac{W}{\sin(90+(30+\theta))} = \frac{P}{\sin(90+60)}$$

$$\frac{W/2}{\cos \theta} = \frac{7W}{\cos \theta}$$

SATB

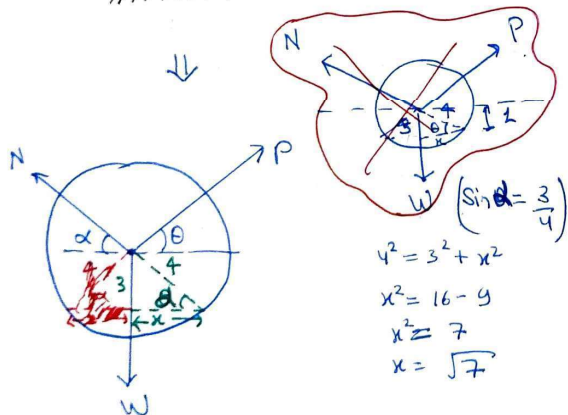
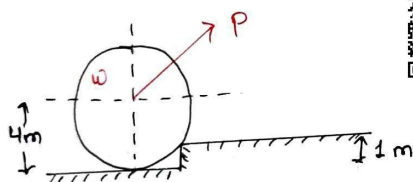
Que (4)! - Compute the force F to just move the roller.



$$\frac{W}{\sin(120 - \theta)} = \frac{F}{\sin(90 + \theta)} = \frac{N}{\sin 150}$$

UN-7

Que (5) Compute the minimum required force 'p' to move the roller?



$$\frac{W}{\sin[120 - (\theta + \alpha)]} = \frac{P}{\sin(90 + \theta)} = \frac{N}{\sin(90 + \theta)}$$

$$P = \frac{W \sin(90 + \theta)}{\sin[120 - (\theta + \alpha)]} = \frac{W \cos \alpha}{\sin(\theta + \alpha)}$$

For P to be minimum $\Rightarrow \sin(\theta + \alpha)$ must be max.

$$\text{Let } y = \sin(\theta + \alpha) = \sin \theta \cos \alpha + \cos \theta \sin \alpha$$

$$= \frac{\sqrt{7}}{5} \sin \theta + \frac{3}{5} \cos \theta$$

For y to be max. $\frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = \frac{\sqrt{7}}{5} \cos \theta - \frac{3}{5} \sin \theta = 0$$

$$= \sqrt{7} \cos \theta - 3 \sin \theta = 0$$

divid by $\cos \theta$

$$\Rightarrow \sqrt{7} - 3 \tan \theta = 0$$

$$\tan \theta = \frac{\sqrt{7}}{3}$$

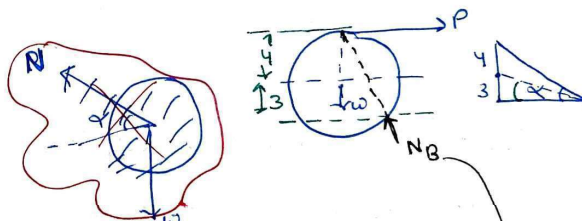
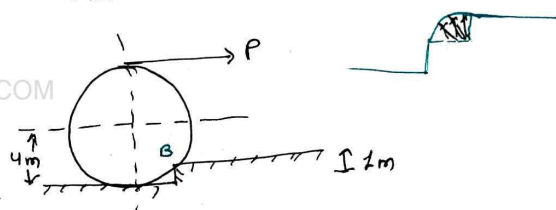
$$\theta = 41.4^\circ$$

$$P'_{\min} = \frac{W \cos \alpha}{\sin(\theta + \alpha)} = \frac{W(\sqrt{7}/5)}{\sin(41.4 + 48.6)}$$

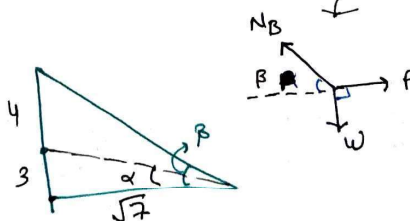
$$P_{\min} = \frac{W(\sqrt{7}/5)}{\sin(90)} = \frac{W\sqrt{7}}{5} \text{ Ans}$$

Imp

Que (6)! - Compute the force required to pull the roller?



For roller to just move, $N_A = 0$
 Since these three forces (P, N_B, W) are coplanar, non-linear, radial and are keeping the roller in equilibrium hence these forces must be concurrent.



$$\cot B = \frac{\sqrt{7}}{7} = 69.3^\circ$$

$$\frac{P}{\sin(90 + \beta)} = \frac{W}{\sin(180 - \beta)} = \frac{N_B}{\sin 90}$$

Moment of Force:-

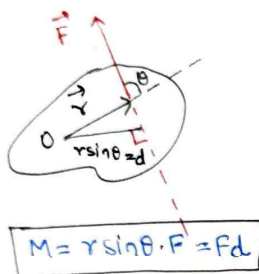
It is the turning effect produced by a force, on the body on which it acts.

The moment of force is equal to the product of the force and perpendicular distance of the point, about which moment is required and line of action of force.

$$\vec{M} = \vec{r} \times \vec{F}$$

$$= rF \sin \theta \hat{u}$$

$\hat{u}$ = unit vector in plane perpendicular to $\vec{r}$ and $\vec{F}$ and given by Right Hand Thumb rule.



also $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$
 $\vec{r} = r_1 \hat{i} + r_2 \hat{j} + r_3 \hat{k}$
 $\vec{M} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_1 & r_2 & r_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

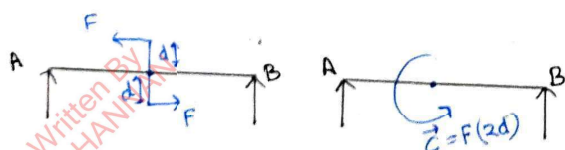
Depending upon the effect of moment on the rotation of the body, moment is classified as:-

- Clockwise moment ↻
- Anticlockwise moment ↺

Couple $\vec{C}$ or (moment of couple) / Free Vector.

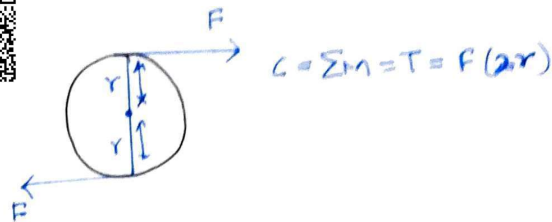
It is equal and opposite forces acts on a body at a distance apart they form a couple.

Since Impact of this couple is same on the body irrespective of point of application it is termed as Free Vector.



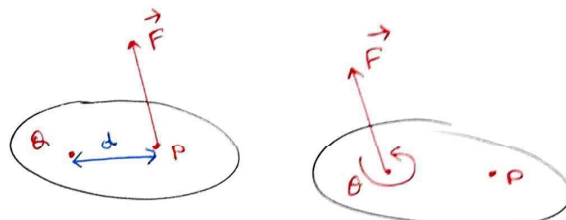
TORQUE:-

It is the moment of force or forces acting on the body about the any axis.



Note:-

If a force is acting on a body at same point "P" and is transferred to same other point "O" away from its line of action, it will be converted in to equivalent force Couple System.



moment at O
 $[M_O = F \cdot d]$

UN-2

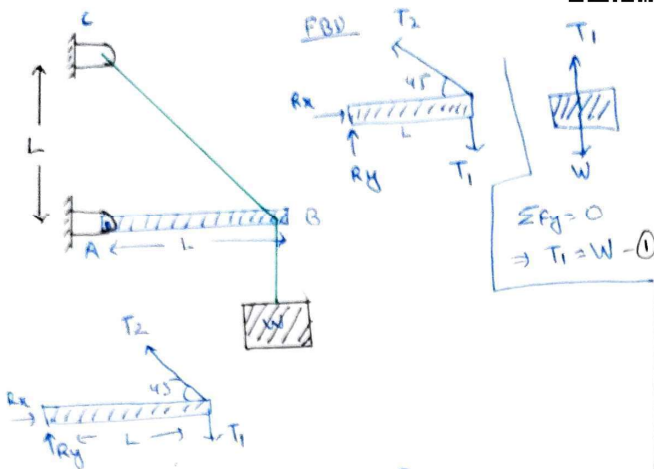
Variignon's Principle of moments / Law of moments.

It states, If a number of coplanar forces are acting simultaneously on a body, the algebraic summation of the moments of all the forces about any point is equal to the moment of their resultant force about the same point.



Question 1-

Compute the reactions at point A, if bar B is massless and supported by a cable BC. A weight (W) is also hanged at B. Find the tension in cable.



$$\begin{aligned} \sum M_A = 0 \\ \Rightarrow T_2 \sin 45^\circ \cdot L = T_1 \cdot L \\ T_2 = \sqrt{2} T_1 \\ T_2 = \sqrt{2} W \end{aligned}$$

$$\sum F_y = 0$$

$$R_y + T_2 \cos 45^\circ = T_1$$

$$R_y = T_1 - \frac{T_2}{\sqrt{2}}$$

$$R_y = W - \frac{\sqrt{2} W}{\sqrt{2}} = 0$$

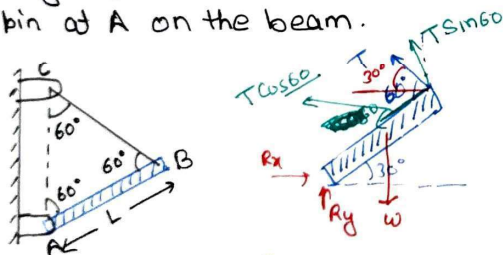
$$\begin{aligned} \sum F_x = 0 \\ R_x = T_2 \cos 45^\circ \\ R_x = \frac{T_2}{\sqrt{2}} \Rightarrow T_2 = R_x \sqrt{2} \\ R_x = \frac{\sqrt{2} W}{\sqrt{2}} = W \end{aligned}$$

Hence:-

$$\begin{aligned} T_1 &= W \\ T_2 &= \sqrt{2} W \\ R_x &= W \\ R_y &= 0 \end{aligned}$$

Que 2:-

An uniform beam AB is pinned at A and is held by cable BC in the position. If the tension in cable is T, then compute the weight of the beam and the reaction of bin at A on the beam.



$$\sum F_x = 0$$

$$R_x = T \cos 30^\circ$$

$$R_x = \frac{\sqrt{3}}{2} T = \frac{\sqrt{3}}{2} \times \frac{W}{2}$$

$$R_x = \frac{\sqrt{3}}{4} W$$

$$\sum F_y = 0$$

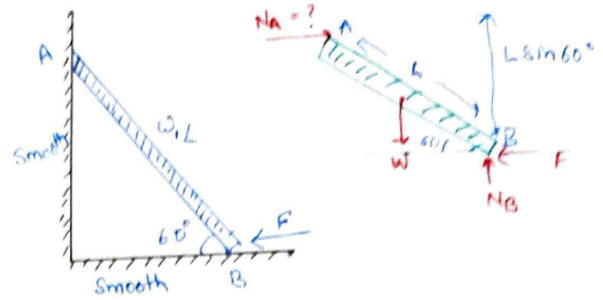
$$R_y + T \sin 30^\circ - W = 0$$

$$R_y = W - \frac{W}{2} = \frac{W}{2}$$

$$R = \sqrt{(R_x)^2 + (R_y)^2} = \sqrt{\left(\frac{\sqrt{3}}{4} W\right)^2 + \left(\frac{W}{2}\right)^2} = \frac{W}{2} \sqrt{3}$$

Que 3:-

Compute the force "F" required to keep the given bar in equilibrium.



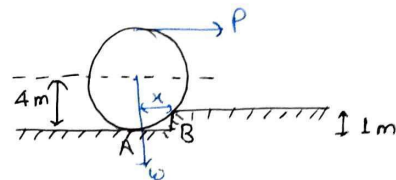
$$\begin{aligned} \sum M_B = 0 \\ N_A \cdot L \sin 60^\circ = W \cdot \frac{L}{2} \cos 60^\circ \\ N_A = \frac{W}{2 \sqrt{3}} \end{aligned}$$

$$\sum F_x = 0$$

$$N_A = F = \frac{W}{2 \sqrt{3}} \text{ Ans}$$

→ From previous Que (6) 2nd method

Que 4 Compute the force Required to pull the roller?



$$\sum M_B = 0$$

$$P \cdot r = W \cdot x$$

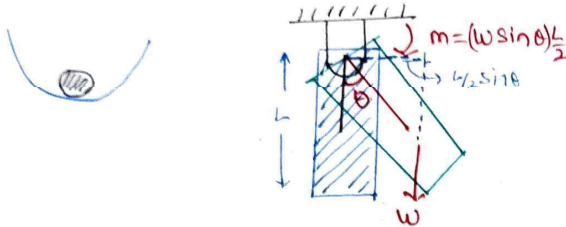
$$P = \frac{W \cdot \sqrt{3}}{7} = \frac{W}{\sqrt{3}} = 0.377 W \text{ Ans}$$

TYPE OF EQUILIBRIUM :-



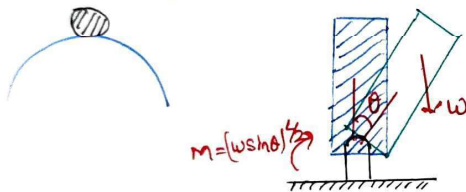
a) Stable Equilibrium :-

When a body is in equilibrium and is slightly displaced from its initial mean position and released, if it comes back to the original position, it is termed as 'Stable Equilibrium'.



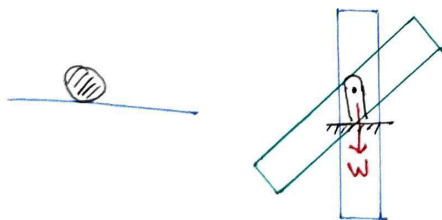
b) Unstable Equilibrium :-

If the body is not able to return back to its initial mean position but further displaces away from initial mean position, it is termed as 'Unstable Equilibrium'.



c) Neutral Equilibrium :-

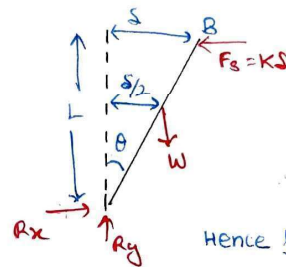
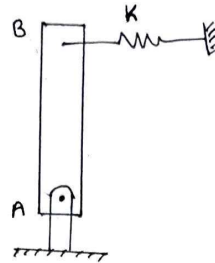
If the body is not able to return back to its initial mean position but again attains equilibrium at new displaced position, it is termed as 'Neutral Equilibrium'.



Written By
A. HANNAN

Ques 1

A bar AB of length 'L' and weight 'w' is hinged at 'A' and is supported by a spring of stiffness 'K' as shown. For stable equilibrium of the system find the value of 'K'.



$$\begin{aligned}\Sigma M_A &= 0 \\ F_s \cdot L &= W \times \frac{L}{2} \\ Ks \cdot L &= W \times \frac{L}{2}\end{aligned}$$

$$\left[K = \frac{W}{2L} \right]$$

Hence for stable equilibrium

$$K > \frac{W}{2L}$$

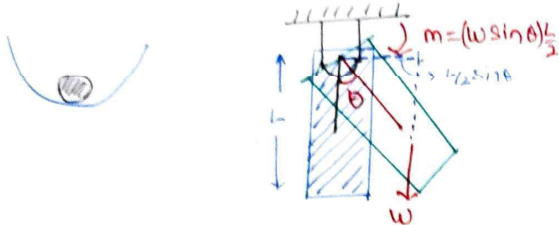
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TYPE OF EQUILLIBRIUM :-



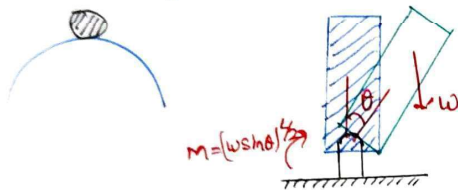
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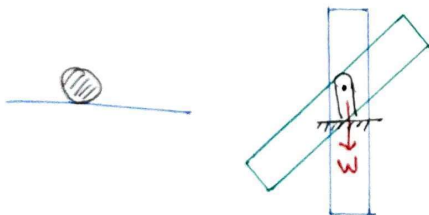
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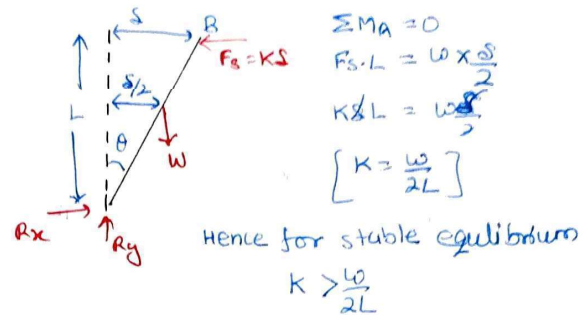
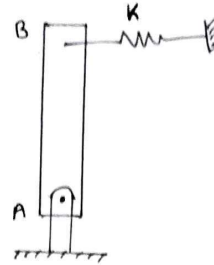
If the body is not able to return back to its initial mean position but again attains equilibrium at new displaced position, it is termed as 'Neutral Equilibrium'.



Written By
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Ques 1

A bar AB of length 'L' and weight 'w' is hinged 'A' and is supported by a spring of stiffness 'K' as shown. For stable equilibrium of the system the find the value of "K".



$$\begin{aligned}\Sigma M_A &= 0 \\ F_s \cdot L &= W \times \frac{L}{2} \\ Ks \cdot L &= W \times \frac{L}{2} \\ \left[K = \frac{W}{2L} \right]\end{aligned}$$

Hence for stable equilibrium

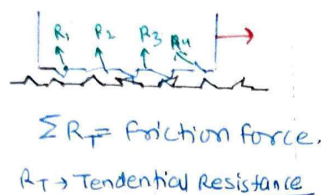
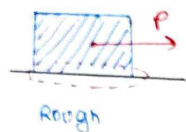
$$K > \frac{W}{2L}$$

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FRICTION: -

Due to roughness and irregularities over the surface of the body, when it moved on another surface, a certain degree of interlocking b/w the particles takes place, which resist the movement of the block over the surface.

This force developed due to the interlocking between the particles is termed as FRICTION.



This friction is further classified into:-

- A) Static Friction
- B) Dynamic Friction
 - i) Sliding Friction
 - ii) Rolling Friction

Limiting Friction.

① Static Friction:-

It is the Friction experienced by the body when it is at rest or when the body is on the verge of movement (Impending moment)

⑤ Dynamic friction :-

It is the friction experienced by the body when it is in motion.

-It is further of two types -

① Sliding friction:-

Sliding Friction:
It is the friction experienced by body when it slides over another body.

(ii) Rolling Friction! —

It is the friction experienced by body when it ~~not~~ rolls over another body.

NOTE :-

Limiting Friction:-

Limiting Friction:
The maximum value of friction force, which comes into action when a body just begins to slide over the surface of another body is termed as "Limiting Friction".

UN-9

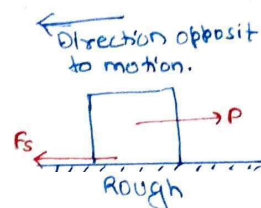
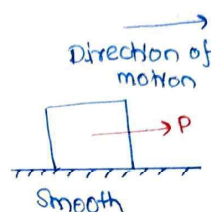
UN-9
Law of Friction! — (Coulomb's Law): —

He proposed following law of friction.

- A) Law of static friction
- B) Law of kinetic/ Dynamic friction

A) Law of Static Friction:-

- a) The force of friction always act in a direction, opposite to that in which body tends to move, if the force of friction would have been absent



- b) The magnitude of force of friction is exactly equal to the force, which tends to move the body
ie $|F_s| = P$

- c) The value of maximum static friction is directly proportional to the normal reaction b/w the surfaces.

$$[F_{s \max} \text{ \& } N]$$

$$F_s = \mu_s N$$

"Normal Force is responsible to break the contact from surface"

μ_s = coefficient of static friction.

- d) The force of friction is independent of area of contact b/w the two surfaces.
- e) The force of friction depends upon the roughness of surface.

B) Law of Kinetic/Dynamic Friction:-

- a) The force of friction always act in the direction opposite to that in which the body is moving.
- b) The value of kinetic friction is also directly proportional to the normal reaction.

$$[F_K \propto N]$$

$$R_K = N_K N$$

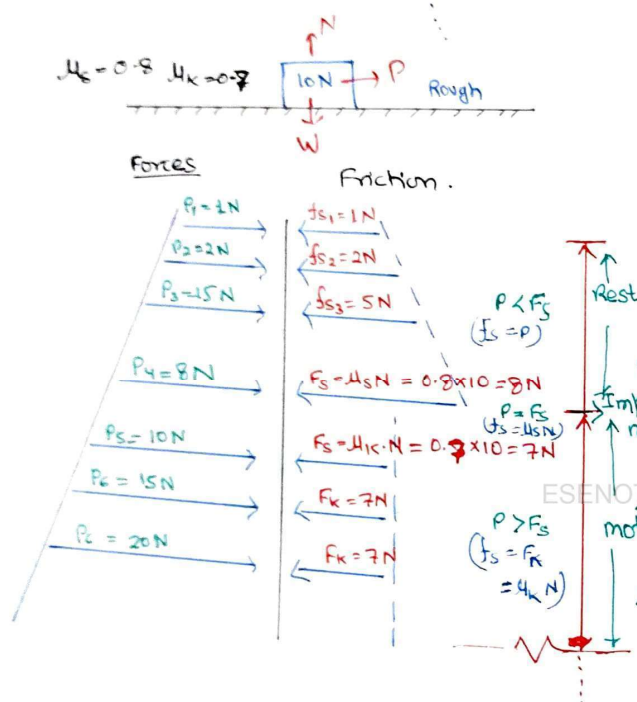
μ_k = Coeff. of kinetic/dynamic friction.

NOTE

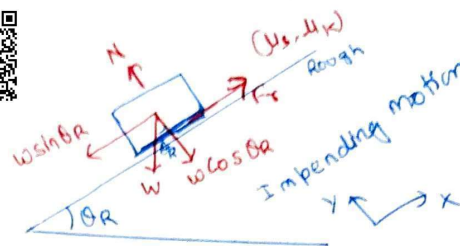
generally value of $\mu_k < \mu_s$ (and μ_k & μ_s is assumed to be constant for all point on the surface)

Generally μ_k , μ_s is different for different point on the surface of the block. But for simplicity, we assume, that μ_k and μ_s are constant for all point on the surface of the block.

c) For moderate speed, this force of friction remain constant but decreases slightly with increases of speed.



Angle of Repose is the angle of the inclined plane at which the block will be just about to slide down.



$$\Sigma F_x = 0 \Rightarrow W \sin \theta = f_s = \mu_s N \quad (i)$$

$$\Sigma F_y = 0 \Rightarrow N = W \cos \theta \quad (ii)$$

from (i) and (ii)

$$W \sin \theta = \mu_s W \cos \theta$$

$$\tan \theta = \mu_s$$

$$\theta = \tan^{-1} \mu_s$$

Angle of Repose.

- If body Rest $< \theta_R \Rightarrow$ Static
- If body at $= \theta_R \Rightarrow$ Impending motion
- If plane $> \theta_R \Rightarrow$ motion start.

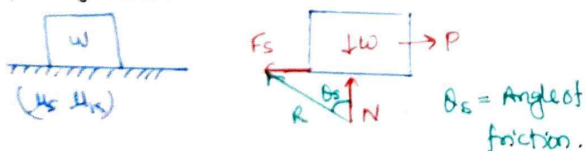
Hence 1-

- * If $\theta < \theta_R \Rightarrow$ Body at rest and $f = f_s$
- * $\theta = \theta_R \Rightarrow$ Body is Impending motion and $f = f_s = \mu_s N$
- * $\theta > \theta_R \Rightarrow$ Body is in motion and $f = f_k = \mu_k N$.

Angle of friction and angle of Repose.

Angle of friction is the angle between the normal axis and resultant of normal and friction force at the point of Impending motion.

Impending motion.



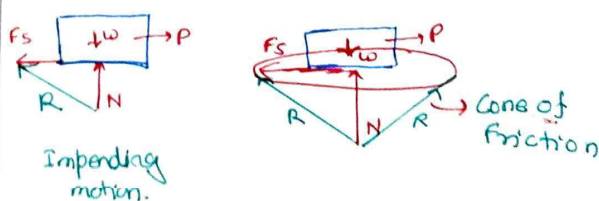
$$\tan \theta_s = \frac{f_s}{N} = \frac{\mu_s N}{N} = \mu_s$$

$$\theta_s = \tan^{-1} (\mu_s)$$

1:07:00

Cone of Friction :-

When the resultant of normal reaction and friction force is revolved around the vertical axis, a cone is formed termed "Cone of Friction"



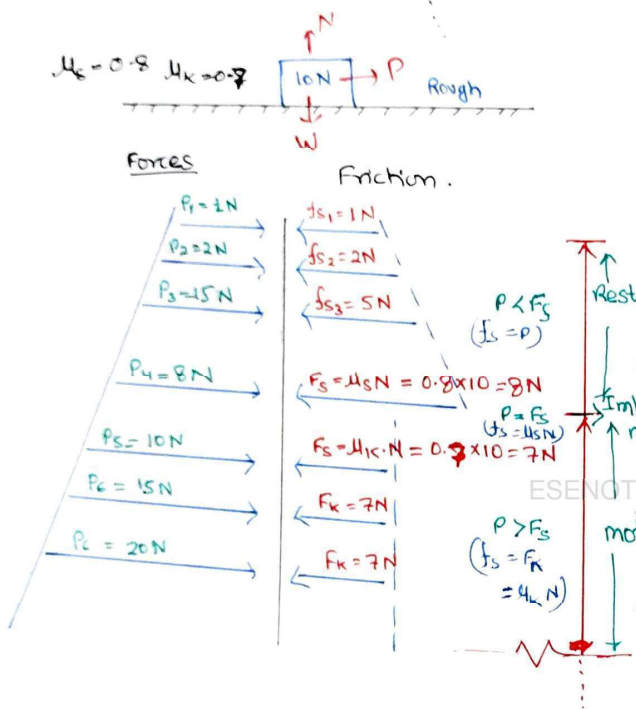
- $\Rightarrow$ If Resultant force lies inside the cone surface $\Rightarrow$ Rest
- $\Rightarrow$ If Resultant force lies on the cone surface $\Rightarrow$ Impending motion.
- $\Rightarrow$ Resultant force never lies outside the cone of friction.

NOTE

generally value of $\mu_k < \mu_s$ (and $\mu_k \neq \mu_s$ is assumed to be constant for all point on the surface)

Generally μ_k, μ_s at different points, μ_k and μ_s are different value surface at different point. But for simplicity, we assume, that μ_k and μ_s are value surface at different point constant (constant in time and space).

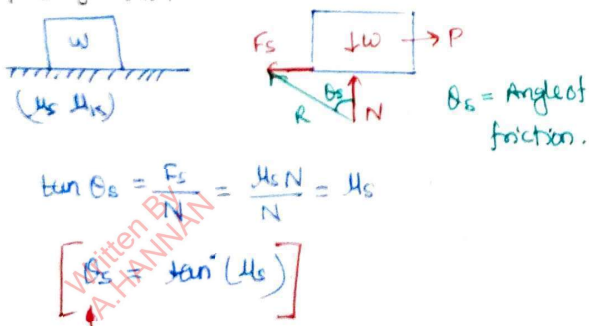
c) For moderate speed, this force of friction remain constant but decreases slightly with increases of speed.



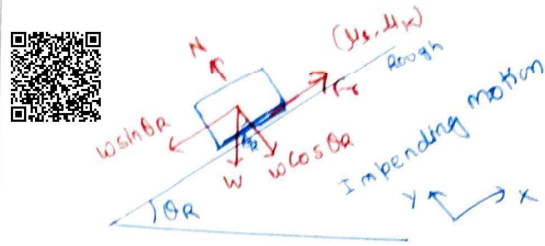
Angle of Friction and angle of Repose.

Angle of friction is the angle between the normal axis and resultant of normal and friction force at the point of impending motion.

Impending motion.



Angle of Repose is the angle of the inclined plane at which the block will be just about to slide down.



$$\Sigma F_x = 0 \rightarrow W \sin \theta_R = f_s = \mu_s N \quad (i)$$

$$\Sigma F_y = 0 \rightarrow N = W \cos \theta_R \quad (ii)$$

from (i) and (ii)

$$W \sin \theta_R = \mu_s W \cos \theta_R$$

$$\tan \theta_R = \mu_s$$

$$\theta_R = \tan^{-1} \mu_s$$

Angle of Repose.

- If body Rest $< \theta_R \Rightarrow$ static
- If body at $= \theta_R \Rightarrow$ Impending motion
- If plane $> \theta_R \Rightarrow$ motion start.

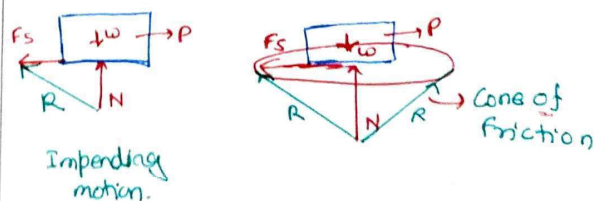
Hence 1 -

- * If $\theta < \theta_R \Rightarrow$ Body at rest and $f = f_s$
- * $\theta = \theta_R \Rightarrow$ Body is Impending motion and $f = f_s = \mu_s N$
- * $\theta > \theta_R \Rightarrow$ Body is in motion and $f = f_k = \mu_k N$.

1:07:00

Cone of Friction :-

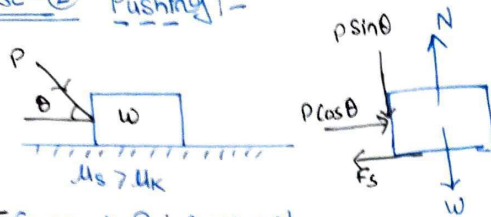
when the resultant of normal reaction and friction force is revolved around the vertical axis, a cone is formed termed "Cone of Friction"



- $\Rightarrow$ If resultant force lies inside the cone surface $\Rightarrow$ Rest
- $\Rightarrow$ If resultant force lies on the cone surface $\Rightarrow$ Impending motion.
- $\Rightarrow$ Resultant force never lies outside the cone of friction.

Ques! - Analyse whether it is easy to push or pull a block on rough surface.

Case - ① Pushing -



$$\Sigma F_y = 0 \Rightarrow P \sin \theta + W = N$$

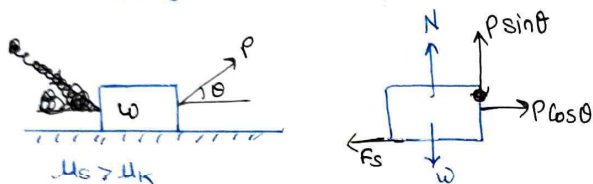
$$\Sigma F_x = 0 \Rightarrow P \cos \theta = F_s = \mu_s N$$

$$P \cos \theta = \mu_s (P \sin \theta + W)$$

$$P (\cos \theta - \mu_s \sin \theta) = \mu_s W$$

$$P = \frac{\mu_s W}{\cos \theta - \mu_s \sin \theta}$$

Case ② Pulling :-



$$\Sigma F_y = 0 \Rightarrow N + P \sin \theta = W$$

$$\Rightarrow N = W - P \sin \theta$$

$$\Sigma F_x = 0 \Rightarrow P \cos \theta = F_s = \mu_s N$$

$$\Rightarrow P \cos \theta = \mu_s (W - P \sin \theta)$$

$$\Rightarrow P (\cos \theta + \mu_s \sin \theta) = \mu_s W$$

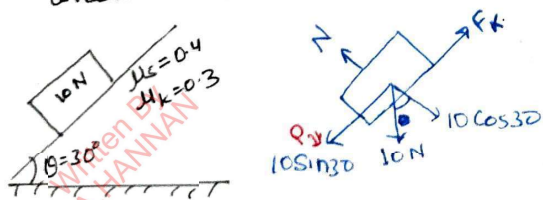
$$\Rightarrow P = \frac{\mu_s W}{\cos \theta + \mu_s \sin \theta}$$

Conclusion :-

$$\because \frac{\mu_s W}{\cos \theta - \mu_s \sin \theta} > \frac{\mu_s W}{\cos \theta + \mu_s \sin \theta}$$

$\therefore$ Hence Less Force Required to displace the body by Pulling.

Imp:
Ques ② Find the state of block and also compute the nature and value and direction of friction.



$$\theta_a = \tan^{-1}(0.4) \quad \therefore \theta > \theta_a \Rightarrow \text{Body is in motion}$$

$$\theta_a = 21.8^\circ$$

$$\Sigma F_x = 0$$

$$\Rightarrow P = 10 \sin 30 = 10 \times \frac{1}{2} = 5N$$

$$\Sigma F_y = 0$$

$$N = 10 \cos 30 = \frac{\sqrt{3}}{2} \times 10 = 5\sqrt{3}N$$

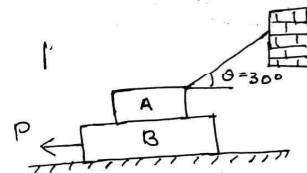
$$F_s = \mu_s N = (0.4 \times 5\sqrt{3}) = 3.464N$$

$$F_k = \mu_k N = (0.3 \times 5\sqrt{3}) = 2.598N$$

Since $P > F_s \Rightarrow$ Friction = Kinetic friction
Nature $= F_k = 2.598N$.

Ques ③ Two blocks A and B of weight 2kN and 4kN respectively are in equilibrium position as shown. If the coefficient of friction b/w the two blocks as well as the block B and floor is 0.3.

Compute the force Required to move the block B.



$$\Sigma F_y = 0$$

$$\Rightarrow N_B = W_B + N_A$$

$$\Rightarrow N_B = 4 + \frac{2\sqrt{3}}{0.3 + \sqrt{3}} = 5.7$$

$$\Sigma F_x = 0$$

$$\Rightarrow P = F_A + F_B$$

$$= \left(\frac{0.3 \times 2\sqrt{3}}{0.3 + \sqrt{3}} \right) + \mu_s N_B$$

$$P = \left(\frac{0.3 \times 2\sqrt{3}}{0.3 + \sqrt{3}} \right) + \left[0.3 \left(\frac{4 + 2\sqrt{3}}{0.3 + \sqrt{3}} \right) \right]$$

$$P = 0.2557 + 1.711$$

$$P = 1.96 = 2$$

$$P = 0.51 + \mu_s N_B$$

$$= 0.51 + (0.3 \times 5.7)$$

$$P = 2.211N \text{ Ans}$$

$$\Sigma F_x = 0$$

$$\Rightarrow N_A + T \sin 30 = W_A$$

$$\Rightarrow T \sin 30 = W_A - N_A$$

$$= T \sin 30 = 2 - N_A \quad \text{--- (i)}$$

$$\Sigma F_x = 0$$

$$T \cos 30 = F_{sA} = \mu_s N_A$$

$$T \cos 30 = 0.3 N_A \quad \text{--- (ii)}$$

from (i) and (ii)

$$\tan 30 = \frac{2 - N_A}{0.3 N_A}$$

$$\frac{1}{\sqrt{3}} = \frac{2 - N_A}{0.3 N_A}$$

$$0.3 N_A = 2\sqrt{3} - \sqrt{3} N_A$$

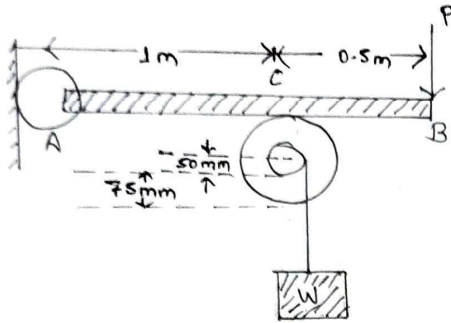
$$N_A = \left(\frac{2\sqrt{3}}{0.3 + \sqrt{3}} \right) = 1.7$$

$$F_{sA} = \mu_s N_A$$

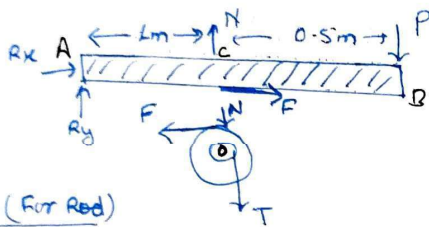
$$F_{sA} = 0.3 \left(\frac{2\sqrt{3}}{0.3 + \sqrt{3}} \right)$$

$$F_{sA} = 0.51$$

Ques ① What is the maximum load (W) which a force 'P' of 5 kN will hold up, if the coefficient of friction at C is 0.2 in arrangement shown. Neglect other friction and weight of the member.



FBD



$\sum M_A = 0$ (For Rod)

$$\Rightarrow N \times 1 = P \times 1.5$$

$$\Rightarrow N = 5 \times 1.5$$

$$N = 7.5 \text{ kN.}$$

$\sum M_D = 0$ (For Pulley)

$$F_s \times 50 = T \times 50$$

$$4N \times 75 = T \times 50$$

$$T = \frac{4N \times 75}{50}$$

$$T = \frac{0.2 \times 7.5 \times 75}{50}$$

$$T = 2.25 \text{ kN.}$$

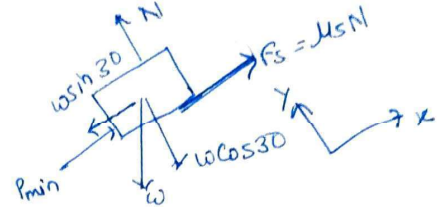
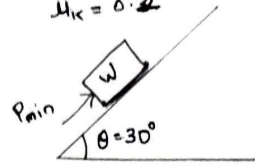
$\sum M_y = 0$ (For Block)

$$T = W = 2.25 \text{ kN.}$$

Ques ② Find the force $P_{\min}$ to just support the block

$$\mu_s = 0.3$$

$$\mu_k = 0.2$$



$\sum F_y = 0$

$$\Rightarrow N = W \cos 30 = \frac{\sqrt{3}W}{2}$$

$\sum F_x = 0$

$$P_{\min} + F_s = W \sin 30$$

$$P_{\min} = \frac{W}{2} - F_s$$

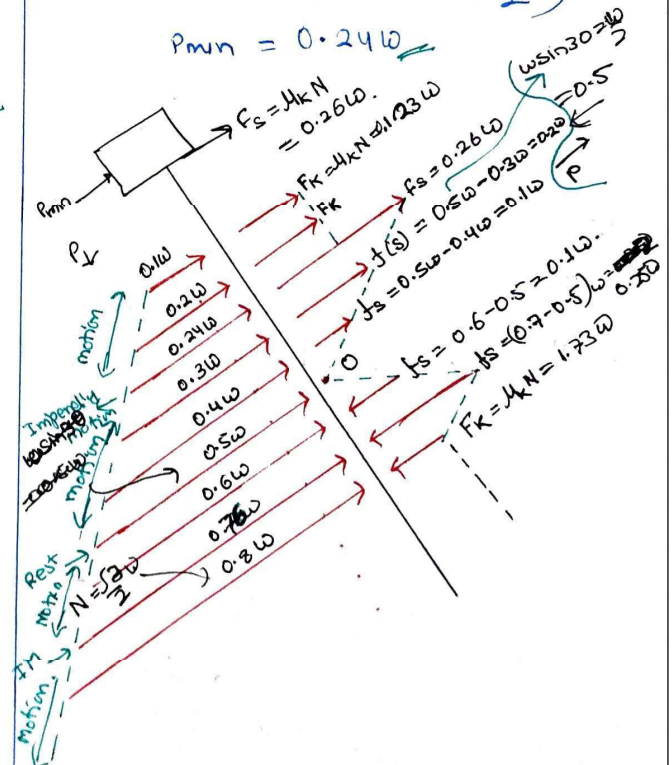
$$P_{\min} = \frac{W}{2} - \mu_s N$$

$$P_{\min} = \frac{W}{2} - 0.3 \times \frac{\sqrt{3}W}{2}$$

$$= W \left(\frac{1}{2} - 0.3 \times \frac{\sqrt{3}}{2} \right)$$

$$= W (0.5 - 0.3 \times 1.735)$$

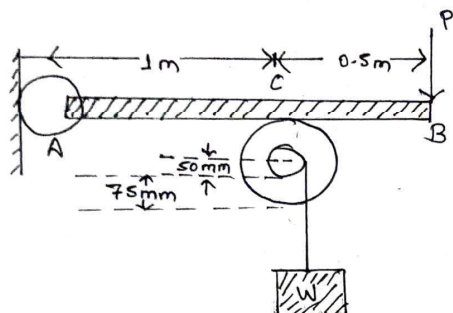
$$P_{\min} = 0.24W$$



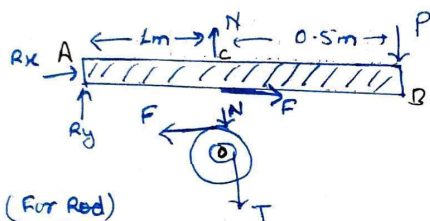
Written By
A.HANNAN



Q1) What is the maximum load (W) which a force ' P ' of 5 kN will hold up, if the coefficient of friction at C is 0.2 in arrangement shown. Neglect other friction and weight of the member.



FBV



$\sum M_A = 0$ (For Rod)

$$\Rightarrow N \times 1 = P \times 1.5$$

$$\Rightarrow N = 5 \times 1.5$$

$$N = 7.5 \text{ kN.}$$

$\sum M_O = 0$ (For Pulley)

$$F \times 75 = T \times 50$$

$$4N \times 75 = T \times 50$$

$$T = \frac{4N \times 75}{50}$$

$$T = \frac{0.2 \times 7.5 \times 75}{50}$$

$$T = 2.25 \text{ kN.}$$

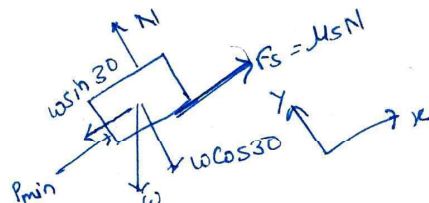
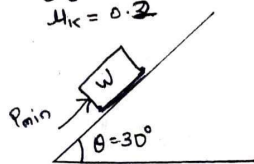
$\sum M_y = 0$ (For Block)

$$T = W = 2.25 \text{ kN.}$$

Q2) Find the force $P_{\min}$ to just support the block

$$\mu_s = 0.3$$

$$\mu_k = 0.2$$



$\sum F_y = 0$

$$\Rightarrow N = W \cos 30 = \frac{\sqrt{3}W}{2}$$

$\sum F_x = 0$

$$P_{\min} + F_s = W \sin 30$$

$$P_{\min} = \frac{W}{2} - F_s$$

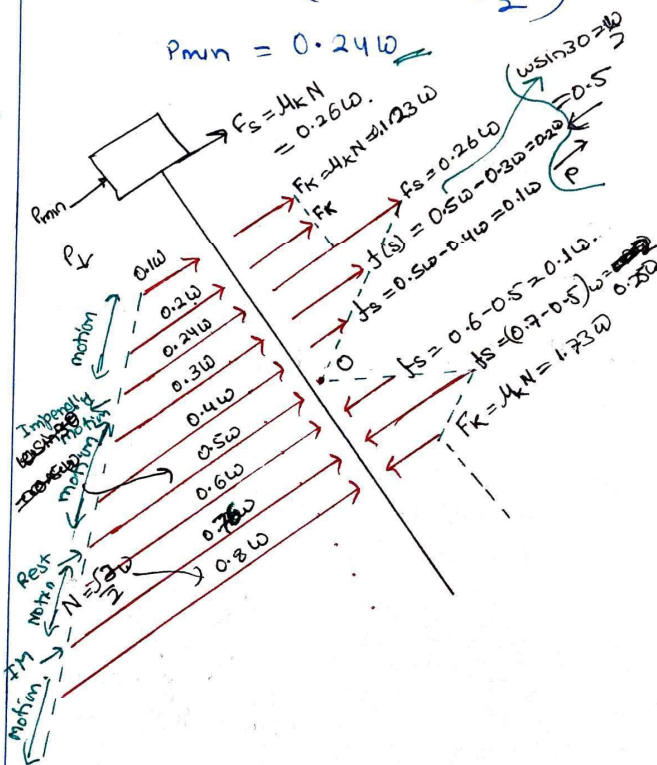
$$P_{\min} = \frac{W}{2} - \mu_s N$$

$$P_{\min} = \frac{W}{2} - 0.3 \times \frac{\sqrt{3}W}{2}$$

$$= W \left(\frac{1}{2} - 0.3 \times \frac{\sqrt{3}}{2} \right)$$

$$= W (0.5 - 0.3 \times 1.735)$$

$$P_{\min} = 0.24W$$



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Que 3 Compute the P_{max} to keep the block in equilibrium in previous problem.

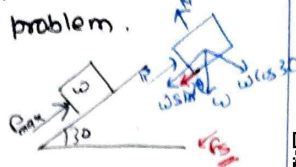
Ans

$$P_{max} = F_s + W \sin 30$$

$$= \mu_s N + \frac{W}{2}$$

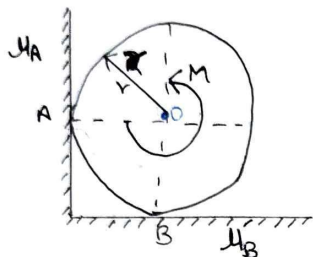
$$= 0.3 \times \frac{\sqrt{3}W}{2} + \frac{W}{2}$$

$$P_{max} = 0.76W \text{ Ans}$$



P_{max} means Block move upward.

Que 4 - Find the Anti clock wise moment "M" applied about the center 'O' to just rotate the cylinder of weight 'W' and radius 'r'.



$$\sum F_x = 0$$

$$\Rightarrow N_A = \mu_B N_B$$

$$\sum F_y = 0$$

$$\Rightarrow N_B + \mu_A N_A - W = 0$$

$$N_B = W - \mu_A N_A$$

$$N_B = W - \mu_A \mu_B N_B$$

$$N_B (1 + \mu_A \mu_B) = W$$

$$\left(N_B = \frac{W}{1 + \mu_A \mu_B} \right)$$

$$\left(N_A = \mu_B N_B = \frac{\mu_B W}{1 + \mu_A \mu_B} \right)$$

$$\sum M_O = 0$$

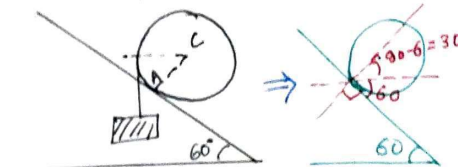
$$\Rightarrow \mu_A N_A r + \mu_B N_B r = M$$

$$\Rightarrow M = \left[\frac{\mu_A \mu_B W}{1 + \mu_A \mu_B} + \frac{\mu_B W}{1 + \mu_A \mu_B} \right]$$

$$\Rightarrow M = r \left[\frac{\mu_A W \mu_B}{1 + \mu_A \mu_B} + \frac{\mu_B W}{1 + \mu_A \mu_B} \right] \text{ Ans}$$

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Que 5 - Compute the tension in string and the minimum weight of block to keep the system just in equilibrium, also find the required coefficient of friction for the plane.



For Block 1 -

$$\sum F_y = 0$$

$$T = W_2$$

For Cylinder

$$\sum F_x = 0$$

$$\mu_s N \cos 60^\circ = N \cos 30^\circ$$

$$\mu_s = \frac{\cos 30^\circ}{\cos 60^\circ}$$

$$\mu_s = \frac{\sqrt{3}}{2} \times \frac{2}{1}$$

$$\mu_s = \sqrt{3} = 1.732$$

$$\sum F_y = 0$$

$$W_1 + T = \mu_s N \sin 60^\circ + N \sin 30^\circ$$

$$W_1 + T = \sqrt{3} N \frac{\sqrt{3}}{2} + \frac{N}{2} = \frac{3N}{2} + \frac{N}{2} = 2N$$

$$W_1 + T = 2N \quad \therefore T = W_2$$

$$W_1 + W_2 = 2N \quad \text{--- (1)}$$

$$\sum M_O = 0$$

$$\mu_s N \sin 60^\circ \cdot r = T \cdot r$$

$$\mu_s N = T = W_2 \quad \text{--- (2)}$$

$$\therefore T = \mu_s N = W_2 \Rightarrow N = \frac{W_2}{\mu_s}$$

from eq (1)

$$W_1 + \mu_s N = 2N$$

$$W_1 + W_2 = 2N = \frac{2W_2}{\mu_s}$$

$$W_1 = \frac{2W_2}{\mu_s} - W_2 = \frac{2\mu_s W_2 - W_2 \mu_s}{\mu_s}$$

$$W_1 =$$

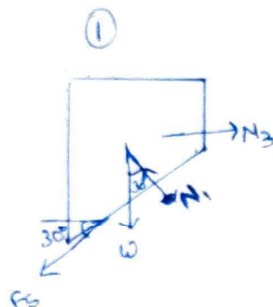
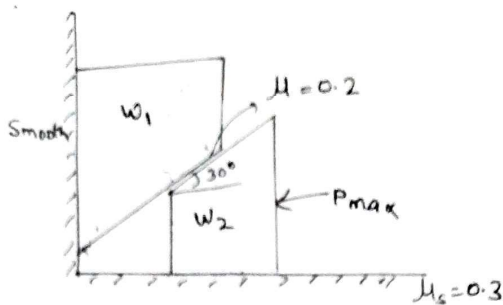
$$\frac{2W_2}{\mu_s} = W_2 = W_1$$

$$W_2 \left(\frac{2 - \mu_s}{\mu_s} \right) = W_1$$

$$W_2 = \frac{W_1 \mu_s}{2 - \mu_s} = \left(\frac{W_1 \sqrt{3}}{2 - \sqrt{3}} \right)$$

$$(W_2 = T = \frac{W_1 \sqrt{3}}{2 - \sqrt{3}}) \text{ Ans}$$

Ques 6:- Find the maximum force 'P' for equilibrium of the system.



Block 1
 $\Sigma F_x = 0$

$$N_3 = N_1 \sin 30^\circ + F_3 \cos 30^\circ$$

$$= \frac{N_1}{2} + \mu_s N_1 \frac{\sqrt{3}}{2}$$

$$N_3 = \frac{N_1}{2} + 0.2 \cdot \frac{\sqrt{3}}{2} N_1$$

$$N_3 = 0.673 N_1$$

$\Sigma F_y = 0$

$$N_1 \cos 30^\circ = W_1 + \mu_s N_1 \sin 30^\circ$$

$$\frac{\sqrt{3}}{2} N_1 = W_1 + \frac{0.2}{2} N_1$$

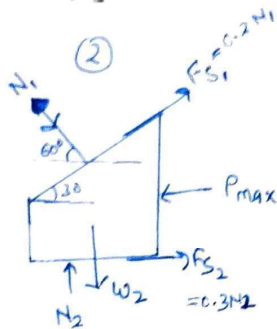
$$\frac{\sqrt{3}}{2} N_1 - \frac{0.2}{2} N_1 = W_1$$

$$\left(\frac{1.732 - 0.2}{2} \right) N_1 = W_1$$

$$\left(\frac{1.532}{2} \right) N_1 = W_1$$

$$W_1 = 0.766 N_1$$

$$N_1 = 1.305 W_1$$



Block 2
 $\Sigma F_x = 0$

$$N_1 \cos 60^\circ + 0.3 N_2 - P_{\max} + 0.2 N_1 \sin 30^\circ = 0$$

$$P_{\max} = \frac{N_1}{2} + 0.3 N_2 + 0.2 N_1 \frac{\sqrt{3}}{2}$$

Block 2

$\Sigma F_x = 0$

$$N_1 \cos 60^\circ + \mu_s N_1 \cos 30^\circ + \mu_s N_2 = P_{\max}$$

$$\frac{1.305 W_1}{2} + \frac{0.2 \cdot \sqrt{3} N_1}{2} + 0.3 N_2 = P_{\max}$$

$$(\because N_1 = 1.305 W_1)$$

$$\frac{1.305 W_1}{2} + \frac{0.2 \cdot \sqrt{3} \cdot 1.305 W_1}{2} + 0.3 N_2 = P_{\max}$$

$$\Rightarrow 0.65 W_1 + 0.226 W_1 + 0.3 N_2 = P_{\max}$$

$$\Rightarrow 0.876 W_1 + 0.3 N_2 = P_{\max} \quad \text{--- (A)}$$

$\Sigma F_y = 0$

$$N_1 \sin 60^\circ + W_2 - N_2 - 0.2 N_1 \sin 30^\circ = 0$$

$$\left(\frac{1.305 \times \sqrt{3}}{2} \right) W_1 + W_2 - N_2 - 0.2 \times 1.305 W_1 \times \frac{1}{2} = 0$$

$$\Rightarrow W_1 \cdot 1.13 + W_2 - N_2 - 0.1305 W_1 = 0$$

$$W_2 - N_2 = 0.1305 W_1 - 1.13 W_1$$

$$W_2 - N_2 = -0.999 W_1$$

$$N_2 = W_2 + 0.999 W_1 \quad \text{--- (B)}$$

$$P_{\max} = 0.876 W_1 + 0.3 N_2$$

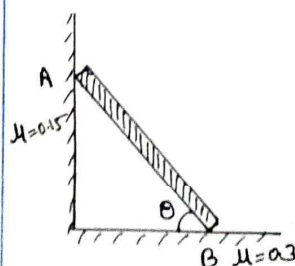
$$P_{\max} = 0.876 W_1 + 0.3 (W_2 + 0.999 W_1)$$

$$= 0.34 W_2$$

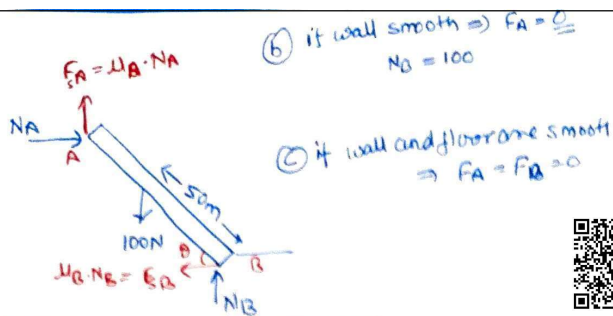
UN-12

Ques 7 A ladder weight 100N and length 100m is just about to slide down, then the inclination 'theta' with horizontal surface.

- Find the change in the value of θ , if wall is smooth.
- Find the change in value of θ , if both wall and floor are smooth.



Written By
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$$\sum F_x = 0$$

$$N_A = \mu_B N_B$$

$$N_A = 0.3 N_B$$

$$N_A = 0.3 \times 95.69$$

$$N_A = 28.708 \text{ N}$$

$$\sum F_y = 0$$

$$N_B + \mu_A N_A = 100$$

$$N_B + \mu(0.3 N_B) = 100$$

$$N_B(1 + 0.15 \times 0.3) = 100$$

$$N_B = 95.69 \text{ N}$$

$$\sum M_A = 0$$

$$F_B \cdot L \sin \theta + 100 \times 50 \cos \theta = N_B \cdot L \cos \theta$$

$$\mu_B N_B \times 100 \sin \theta + 5000 \cos \theta = 95.69 \times 100 \cos \theta$$

$$0.3 \times 95.69 \times 100 \sin \theta + 5000 \cos \theta = 95.69 \times 100 \cos \theta$$

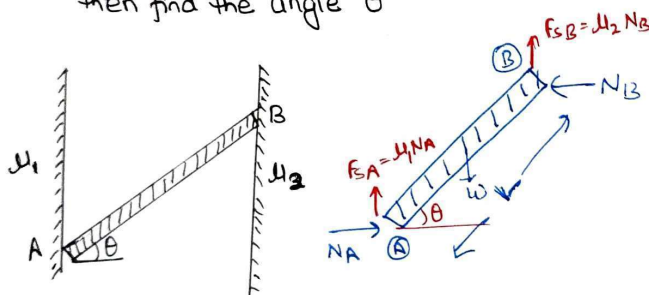
$$2870.7 \sin \theta + 5000 \cos \theta = 9560 \cos \theta$$

$$2870.7 \tan \theta = 9560 - 5000$$

$$\tan \theta = 1.588$$

$$\theta = 57.80^\circ$$

Ques 8 A homogeneous bar AB of weight "w" and length "L" is placed b/w the two vertical plane as shown. If the bar is at rest then find the angle 'θ'



$$\sum F_x = 0$$

$$N_A = N_B$$

$$\sum F_y = 0$$

$$F_B + F_A = w$$

$$\mu_2 N_B + \mu_1 N_A = w$$

$$N_A (\mu_2 + \mu_1) = w \quad (\because N_A = N_B)$$

$$\sum M_B = 0$$

$$\mu_1 N_A L \cos \theta = N_A L \sin \theta + \frac{wL}{2} \cos \theta$$

$$\mu_1 N_A L = N_A L \tan \theta + \frac{wL}{2}$$

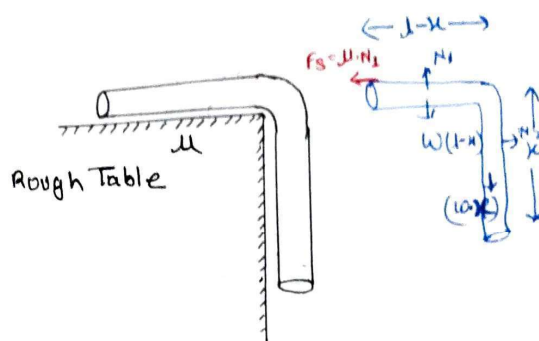
$$N_A L \tan \theta = \mu_1 N_A L - \frac{wL}{2}$$

$$\tan \theta = \frac{2\mu_1 N_A - w}{2N_A}$$

$$\tan \theta = \frac{2\mu_1 N_A - N_A (\mu_2 + \mu_1)}{2N_A}$$

$$\tan \theta = \frac{\mu_1 - \mu_2}{2} \quad \text{Ans}$$

Ques 9 A heavy rope of weight "w" per unit length hanging over an edge of a table. Find the maximum hang length for equilibrium of the rope.



$$wx = F_s = \mu N_1$$

$$\mu x = \mu w(L-x)$$

$$x = \mu L - \mu x$$

$$x + \mu x = \mu L$$

$$x = \frac{\mu L}{1 + \mu} \quad \text{Ans}$$

$$\therefore N_1 = w(L-x)$$

Ques 10 The upper half of an inclined having inclination θ with the horizontal is smooth, while the lower half is rough. If a body of weight "w" slides down from rest at the top, again comes to rest at the bottom of the plane, then determine the value of coefficient of friction for the rough lower half of the plane.

$$F = \text{mass} \cdot \text{accel}$$

If block start from rest

on smooth plane and

comes to rest on rough

half plane, it is possible

when acceleration on smooth plane

is same as retardation on rough plane.

$$\text{Force} = \text{mass} \cdot \text{accel}$$

$$a_s = \frac{\text{force}}{\text{mass}} = \frac{w \sin \theta}{m} = \frac{mg \sin \theta}{m} = g \sin \theta$$

Retardation on Rough Plane :-

Retardation Produced due to friction.

$$r = \frac{\mu_1 N_1 + \mu_2 N_2}{m} = \frac{\mu_1 mg \cos \theta + \mu_2 mg \cos \theta}{m}$$

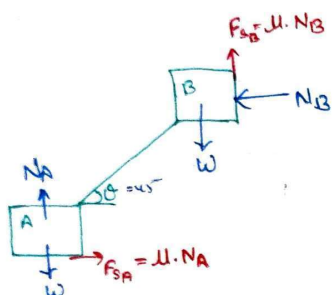
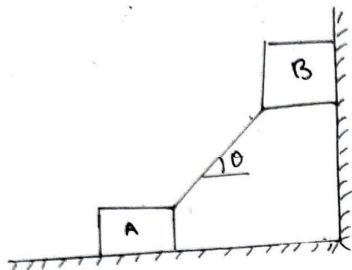
$$r = (g \sin \theta - \mu g \cos \theta)$$

$$g \sin \theta = -g \sin \theta + \mu g \cos \theta$$

$$\mu \cos \theta = 2 \sin \theta$$

$$\mu = 2 \tan \theta \quad \text{Ans}$$

Ques (i) Two identical blocks of weight "w" are supported by a rod inclined at 45° with the horizontal. If both the blocks are in limiting equilibrium, find the coefficient of friction (μ) assuming it to be same at floor as well as at wall.



$$\Sigma F_x = 0$$

$$N_B = \mu \cdot N_A \quad \text{--- (i)}$$

from (i) and (ii)

$$N_B = \frac{\mu \cdot 2w}{1 + \mu^2} \quad \text{--- (iii)}$$

$$\Sigma F_y = 0$$

$$N_A + \mu N_B = 2w$$

$$N_A + \mu(\mu \cdot N_A) = 2w$$

$$N_A(1 + \mu^2) = 2w$$

$$N_A = \left(\frac{2w}{1 + \mu^2} \right) \quad \text{--- (ii)}$$

$$\Sigma M_A = 0$$

$$w \cdot x \cos 45 = N_B \cdot x \sin 45 + \mu \cdot N_B \cdot x \cos 45$$

$$w = N_B + \mu \cdot N_B$$

$$w = \frac{\mu \cdot 2w}{1 + \mu^2} + \frac{\mu \cdot \mu \cdot 2w}{1 + \mu^2}$$

$$1 + \mu^2 = 2\mu + \mu^2$$

$$\Rightarrow \mu^2 + 2\mu - 1 = 0$$

$$\mu = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= (-1 + \sqrt{2}) ; (-1 - \sqrt{2})$$

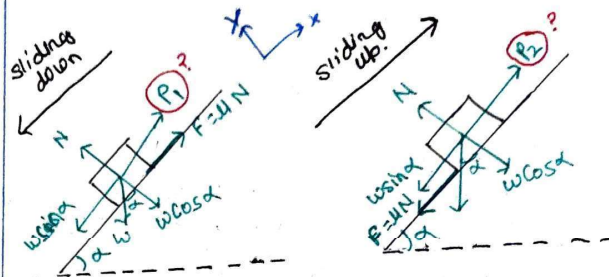
$$\downarrow \quad \downarrow$$

$$0.414 \quad -2.414$$

$$\text{Ans} \quad \times$$

EQUILIBRIUM OF A BODY ON A ROUGH INCLINED PLANE :-

a) when subjected to force acting along the inclined plane.



$$\Sigma F_x = 0$$

$$\Rightarrow P - w \sin \alpha \pm F = 0$$

$$P = w \sin \alpha \mp \mu N = w \sin \alpha \mp \mu w \cos \alpha$$

$$P = w(\sin \alpha \mp \mu \cos \alpha)$$

$$P = w(\sin \alpha \mp \tan \phi \cos \alpha)$$

$$P = w(\sin \alpha \mp \frac{\sin \phi \cos \alpha}{\cos \phi})$$

$$P = \frac{w \sin(\alpha \mp \phi)}{\cos \phi}$$

Hence

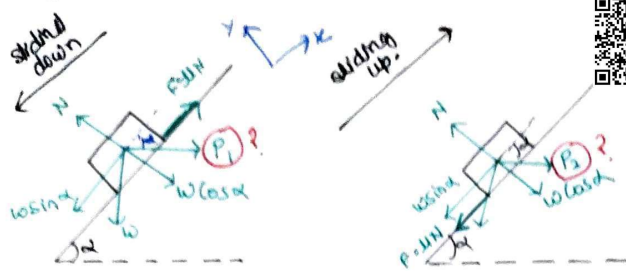
sliding down

$$P_1 = \frac{w \sin(\alpha - \phi)}{\cos \phi}$$

sliding up

$$P_2 = \frac{w \sin(\alpha + \phi)}{\cos \phi}$$

b) Subjected to force acting horizontally:-



$$\sum F_x = 0$$

$$\Rightarrow P \cos \alpha - W \sin \alpha \pm \mu N = 0 \quad \text{--- (i)}$$

$$\sum F_y = 0$$

$$\Rightarrow N = W \cos \alpha + P \sin \alpha \quad \text{--- (ii)}$$

from (i) and (ii)

$$P \cos \alpha - W \sin \alpha \pm \mu (W \cos \alpha + P \sin \alpha) = 0$$

$$P (\cos \alpha \pm \mu \sin \alpha) = W (\sin \alpha \mp \mu \cos \alpha)$$

$$P = \frac{W (\sin \alpha \mp \mu \cos \alpha)}{(\cos \alpha \pm \mu \sin \alpha)} = \frac{W (\sin \alpha \mp \tan \phi \cos \alpha)}{(\cos \alpha \pm \tan \phi \sin \alpha)}$$

$$P = \frac{W (\cos \phi \sin \alpha \mp \sin \phi \cos \alpha)}{(\cos \phi \cos \alpha \pm \sin \phi \sin \alpha)}$$

$$P = \frac{W \sin (\alpha \mp \phi)}{\cos (\alpha \mp \phi)} = W \tan (\alpha \mp \phi)$$

Hence

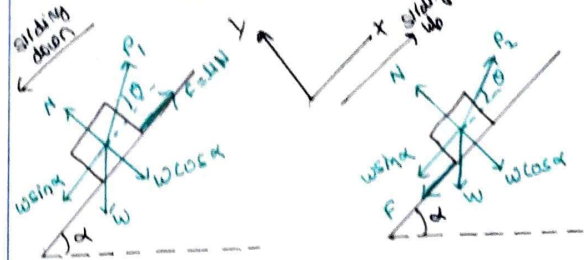
$$P_1 = W \tan (\alpha - \phi)$$

Sliding down

$$P_2 = W \tan (\alpha + \phi)$$

Sliding up

c) When subjected to a force acting at some angle with the inclined plane.



$$\sum F_x = 0$$

$$P \cos \theta - W \sin \alpha \pm F = 0$$

$$P \cos \theta = W \sin \alpha \mp \mu N \quad \text{--- (i)}$$

$$\sum F_y = 0$$

$$P \sin \theta + N - W \cos \alpha = 0$$

$$N = W \cos \alpha - P \sin \theta \quad \text{--- (ii)}$$

from (i) and (ii)

$$P \cos \theta = W \sin \alpha \mp \mu (W \cos \alpha - P \sin \theta)$$

$$P \cos \theta = W \sin \alpha \mp \mu W \cos \alpha \pm \mu P \sin \theta$$

$$P (\cos \theta \mp \mu \sin \theta) = W (\sin \alpha \mp \mu \cos \alpha)$$

$$P = \frac{W (\sin \alpha \mp \mu \cos \alpha)}{(\cos \theta \mp \mu \sin \theta)}$$

$$P = \frac{W \sin (\alpha \mp \phi)}{\cos (\theta \mp \phi)}$$

Hence

Sliding down

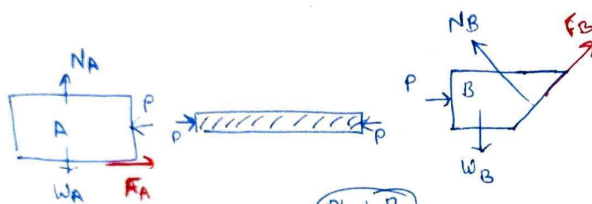
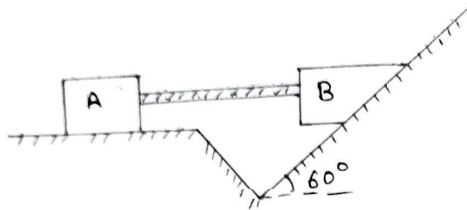
$$P_1 = \frac{W \sin (\alpha - \phi)}{\cos (\theta + \phi)}$$

Sliding up

$$P_2 = \frac{W \sin (\alpha + \phi)}{\cos (\theta - \phi)}$$

"theta is the angle made with inclined plane by P"

Ques (12) Two blocks A and B connected by a horizontal rod and frictionless hinges are supported on two rough planes as shown. The coefficient of friction are 0.3 b/w block 'A' and the horizontal surface and 6.4 b/w block 'B' and the inclined surface. If the block 'B' weights 10N, what is the smallest weight of block 'A', that hold the system in equilibrium.



$$\Sigma F_y = 0$$

$$\Rightarrow N_A = W_A \quad \text{--- (i)}$$

$$\Sigma F_x = 0$$

$$\Rightarrow P = F_A = \mu_s N_A \quad \text{--- (ii)}$$

$$P = 0.3 W_A \quad \text{--- (A)}$$

Block B
Now smallest force required to keep the 'B' in equilibrium is P

P $\rightarrow$ acting // to Horizontal.
2nd case

$$P = W_B \tan(\alpha - \phi) \quad \text{--- (B)}$$

also
 $\tan \phi = \mu_B = 6.4$
 $\Rightarrow \phi = \tan^{-1} 6.4$
 $\phi = 21.8^\circ$

From (A) and (B)

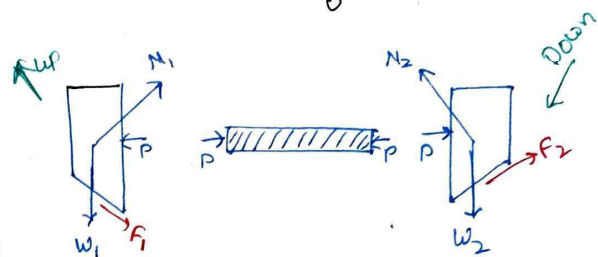
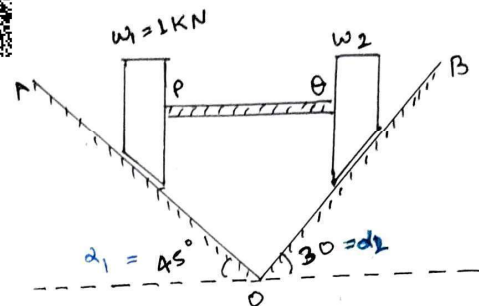
$$P = P$$

$$0.3 W_A = W_B \tan(60 - 21.8^\circ)$$

$$W_B = \frac{0.3 \times 10}{\tan(60 - 21.8^\circ)}$$

Written By W.B = 26.2 N Ans
A.HANNAN

Ques (13) Two loads $w_1 = 1 \text{ kN}$ and w_2 resting on two inclined rough plane are connected by a horizontal link as shown. Find the max. and min values of w_2 for which the equilibrium can exists. consider the angle of friction for both the planes as 20° .



For Equilibrium of load w_2 ,

$$P = w_2 \tan(\alpha_2 - \phi) \quad \text{--- (i)}$$

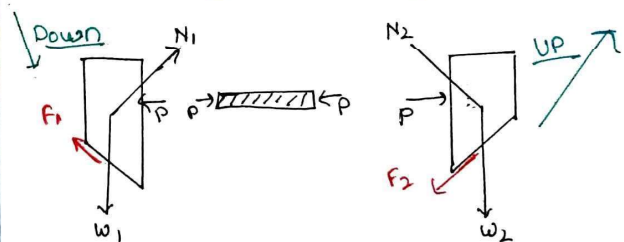
for equilibrium of load w_1 ,

$$P = w_1 \tan(\alpha_1 + \phi) = \text{--- (ii)}$$

From eqⁿ (i) and (ii)

$$w_2 \tan(30 - 20) = 1 \tan(45 + 20)$$

$$\{w_2 = 12.16 \text{ kN}\}$$



For Equilibrium of load w_2 ,

$$P = w_2 \tan(\alpha_2 + \phi) \quad \text{--- (i)}$$

For equilibrium of load w_1 ,

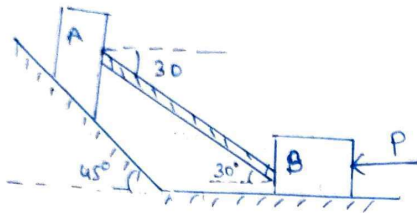
$$P = w_1 \tan(\alpha_1 - \phi)$$

from (i) and (ii)

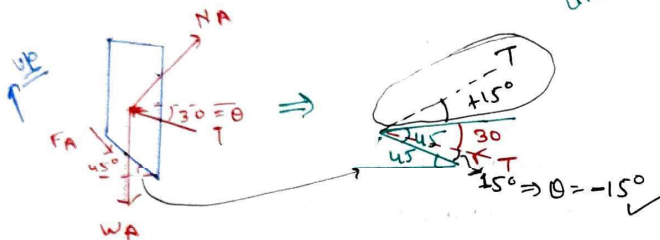
$$w_2 \tan(30 + 20) = 1 \tan(45 - 20)$$

$$\{w_2 = 0.391 \text{ kN}\}$$

Que (14) A block (A) weighing 1 kN rest on a rough inclined plane whose inclination with horizontal is 45° . The block is connected to another block (B) weighing 3 kN resting on a rough horizontal plane by a weightless rigid bar inclined at an angle of 30° to the horizontal as shown. Find the horizontal force (P) required to be applied to the block (B) just to move the block (A) in upward direction. Assume $\phi = 15^\circ$ for all the surface.



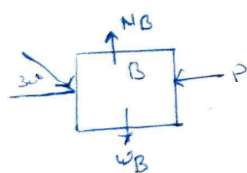
Take care of θ and θ'



For equilibrium of Block A:-

$$T = \frac{W_A \sin(\alpha + \phi)}{\cos(\theta - \phi)} = \frac{1 \sin(45 + 15)}{\cos(-15 - 15)}$$

$$T = 1 \text{ kN}$$



$$\sum F_y = 0$$

$$\Rightarrow N_B - W_B - T \sin 30^\circ = 0$$

$$N_B = 3 + 1 \sin 30^\circ = 3.5 \text{ kN}$$

$$\sum F_x = 0$$

$$P - F_B - T \cos 30^\circ = 0$$

$$P = \mu_B \cdot N_B + T \cos 30^\circ$$

$$P = \tan 15^\circ \times 3.5 + 1 \times \cos 30^\circ$$

$$P = 1.804 \text{ kN}$$

Written By
A. HANNAN

CENTROIDES & MOMENT OF INERTIA

(Objective Point of view)

a) COG: Centre of Gravity:-

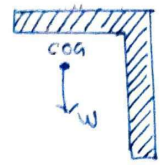
It is the point (inside or outside) of the body where the entire weight of the body is assumed to be concentrated.



Circular Disc



Circular Ring



Angle section.

b) COM: Centre of mass:-

It is the point (inside or outside) of the body where the entire mass of the body is assumed to be concentrated.



NOTE:-

If the gravity is constant throughout the body, in such case COG and COM coincide.

c) CENTROID:-

It is defined as geometrical centre of the body.

- It depends only on the geometry of body (shape and size).

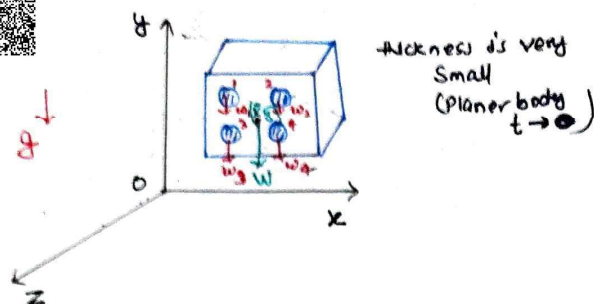
- It does not depend on weight and mass of body

Centroid

CO P	CO L	CO A	CO V
[Center of Point]	[Center of Line]	[Center of Area]	[Center of Volume]
(0-D)	(1-D)	(2-D)	(3-D)

P.T.O

- Centre of gravity can be computed using VANANON'S THEOREM.



$$W\bar{x} = W_1x_1 + W_2x_2 + W_3x_3 \dots W_nx_n$$

$$\bar{x} = \frac{W_1x_1 + W_2x_2 + W_3x_3 \dots W_nx_n}{W}$$

$$\bar{x} = \frac{\sum W_i x_i}{\sum W_i} = \frac{\int x dw}{\int dw}$$

Similarly

$$\bar{y} = \frac{\sum W_i y_i}{\sum W_i} = \frac{\int y dw}{\int dw}$$

Here $[W = W_1 + W_2 + \dots + W_n]$

(Density and thickness assumed to be constant)

Also

$$mg\bar{x} = m_1gx_1 + m_2gx_2 + m_3gx_3 \dots m_ngx_n$$

Here

$$m = m_1 + m_2 + \dots + m_n$$

If $g = \text{constant}$

$$m\bar{x} = m_1x_1 + m_2x_2 + \dots + m_nx_n$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m}$$

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\int x dm}{\int dm} \quad \left(\text{Center of Mass} \right)$$

Similarly

$$\bar{y} = \frac{\sum m_i y_i}{\sum m_i} = \frac{\int y dm}{\int dm}$$

Hence

If $g = \text{constant}$, centre of gravity and centre of mass coincide.

P.T.O

UN-13

$$\text{also if } m\bar{x} = m_1x_1 + m_2x_2 + m_3x_3 \dots m_nx_n$$

$$PV\bar{x} = PV_1x_1 + PV_2x_2 + PV_3x_3 \dots PV_nx_n$$

if $P = \text{constant}$

$$V\bar{x} = V_1x_1 + V_2x_2 + V_3x_3 \dots V_nx_n$$

$$\bar{x} = \frac{V_1x_1 + V_2x_2 + V_3x_3 \dots V_nx_n}{V} \quad (\text{COV})$$

Similarly

$$\bar{y} = \frac{V_1y_1 + V_2y_2 + V_3y_3 \dots V_ny_n}{V}$$

Hence :-

$$\bar{x} = \frac{\sum V_i x_i}{\sum V_i} \quad \bar{y} = \frac{\sum V_i y_i}{\sum V_i}$$

$$\text{OR } \bar{x} = \frac{\int x dv}{\int dv} \quad \text{OR } \bar{y} = \frac{\int y dv}{\int dv}$$

Hence if $P = \text{constant}$ COG, COM, COV all coincide.

also,

$$\text{If } V\bar{x} = V_1x_1 + V_2x_2 + V_3x_3 \dots V_nx_n$$

$$\text{or } A \cdot t \cdot \bar{x} = A_1 \cdot t \cdot x_1 + A_2 \cdot t \cdot x_2 \dots A_n \cdot t \cdot x_n$$

If thickness $(t) = \text{constant}$.

$$A\bar{x} = A_1x_1 + A_2x_2 + A_3x_3 \dots A_nx_n$$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3 \dots A_nx_n}{A} \quad (\text{COA})$$

Hence

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{\int x dA}{\int dA}$$

Similarly

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{\int y dA}{\int dA}$$

If thickness $(t) = \text{constant}$ COV and COA coincide.

$$\text{Also, if } A\bar{x} = A_1x_1 + A_2x_2 + A_3x_3 \dots A_nx_n$$

$$lb\bar{x} = l_1bx_1 + l_2bx_2 + l_3bx_3 \dots l_nbx_n$$

If width $(b) = \text{constant}$.

$$l\bar{x} = l_1x_1 + l_2x_2 + l_3x_3 \dots l_nx_n$$

$$\bar{x} = \frac{l_1x_1 + l_2x_2 + l_3x_3 \dots l_nx_n}{l} \quad (\text{COL})$$

$$\bar{x} = \frac{\sum l_i x_i}{\sum l_i} = \frac{\int x dl}{\int dl}$$

Similarly

$$\bar{y} = \frac{\sum l_i y_i}{\sum l_i} = \frac{\int y dl}{\int dl}$$

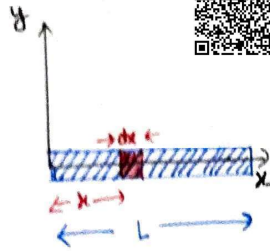
Hence, if width $(b) = \text{constant}$ COA and COL coincide.

[A] Compute the centroid of Rod of Length "L".

$$\bar{x} = \frac{\int_0^L x dx}{\int_0^L dx} = \frac{\left[\frac{x^2}{2} \right]_0^L}{\left[x \right]_0^L}$$

$$\bar{x} = \frac{L^2}{2L} = \frac{L}{2}$$

$$\bar{y} = 0$$



Example 1

Find the centroid of Composite line

$$C_1 = (2, 4)$$

$$C_2 = (4, 2)$$

$$C_3 = (3, 0)$$

$$\bar{x} = \frac{\sum L_i x_i}{\sum L_i}$$

$$= \frac{(4 \times 2) + (4 \times 4) + (2 \times 3)}{4 + 4 + 2}$$

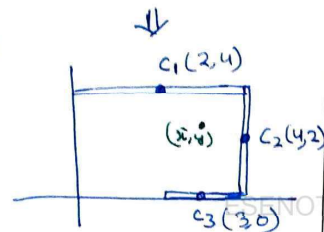
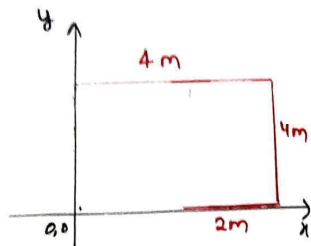
$$\bar{x} = 3m$$

$$\bar{y} = \frac{\sum L_i y_i}{\sum L_i}$$

$$= \frac{(4 \times 4) + (4 \times 2) + (2 \times 0)}{4 + 4 + 2}$$

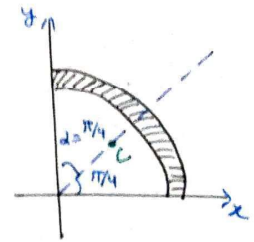
$$\bar{y} = 2.4$$

$$(\bar{x}, \bar{y}) = (3, 2.4)$$



$$(ii) \text{COL} = \left(\frac{2\sqrt{2}R}{\pi} \cos \frac{\pi}{4}, \frac{2\sqrt{2}R}{\pi} \sin \frac{\pi}{4} \right)$$

$$\text{COL} = \left(\frac{2R}{\pi}, \frac{2R}{\pi} \right)$$



[C] Centroid of Rectangular area:-

$$\bar{x} = \frac{\int x dA}{\int dA} = \frac{\int_0^b x \cdot l dx}{\int_0^b l dx}$$

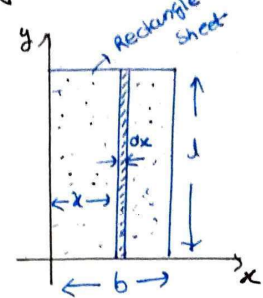
$$= \frac{\left[\frac{x^2}{2} \right]_0^b}{\left[x \right]_0^b}$$

$$= \frac{b^2/2}{b} = \frac{b}{2}$$

$$\left[\bar{x} = \frac{b}{2} \right]$$

Similarly:-

$$\left[\bar{y} = \frac{l}{2} \right]$$



DM-14

[B] Centroid of Circular arc:-

$$\text{COL} = \left(\frac{R \sin \alpha}{\alpha}, 0 \right)$$

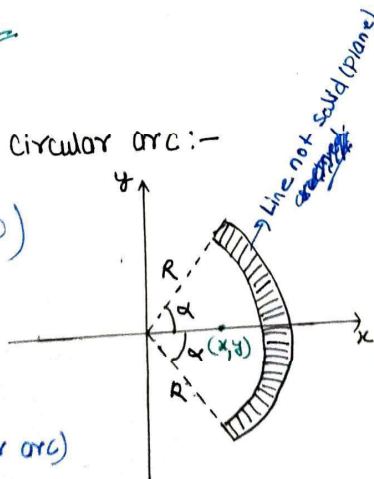
" α " - Radius"

(i) If $\alpha = 90^\circ = \pi/2$
(semicircular arc)

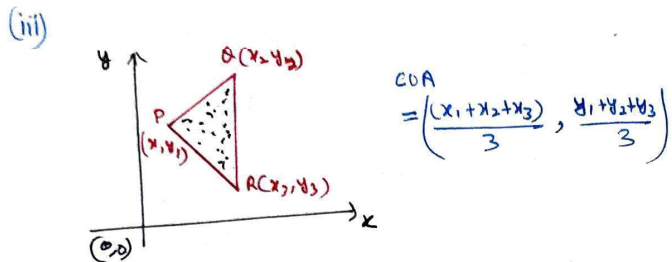
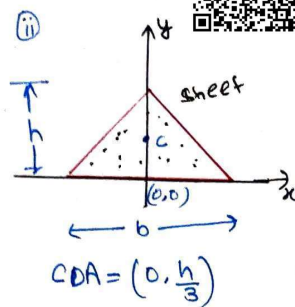
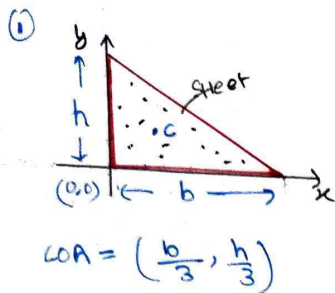
$$\text{COL} = \left(\frac{R \sin(\pi/2)}{\pi/2}, 0 \right) = \left(\frac{2R}{\pi}, 0 \right)$$

(ii) If $\alpha = 45^\circ = \pi/4$

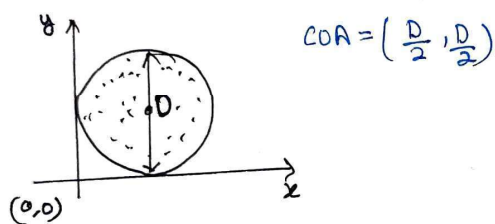
$$\text{COL} = \left(\frac{R \sin(\pi/4)}{\pi/4}, 0 \right) = \left(\frac{2\sqrt{2}R}{\pi}, 0 \right)$$



b) Triangular Area:-

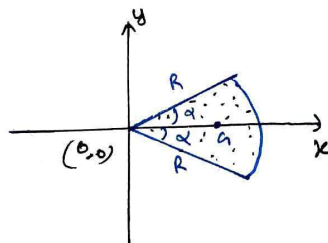


c) Circular Area:-



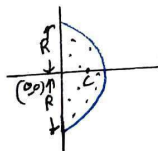
d) Circular Sector:-

COA = $(\frac{2}{3} \frac{R \sin \alpha}{\alpha}, 0)$
 α - Radian.



e) Semi circular Area:-

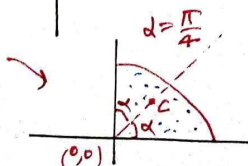
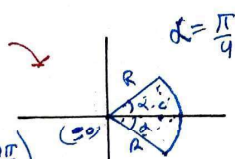
COA = $(\frac{4R}{3\pi}, 0)$



f) Quarter circular Area:-

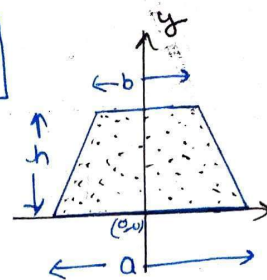
i) COA = $(\frac{4\sqrt{2}R}{3\pi}, 0)$

ii) COA = $(\frac{4\sqrt{2}R \cos \frac{\pi}{4}}{3\pi}, \frac{4\sqrt{2}R \sin \frac{\pi}{4}}{3\pi})$
 $= (\frac{4R}{3\pi}, \frac{4R}{3\pi})$



COA = $(0, \frac{h}{3} (\frac{a+2b}{a+b}))$

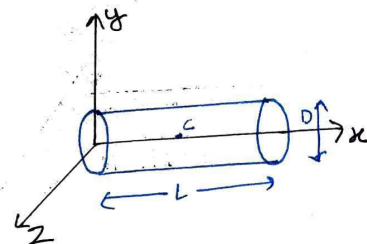
* यदि हमें मध्यम क्षेत्र
 ज्ञात करना है तो
 double करें L $a+2b$



[C] CENTROID OF VOLUME (COV):-

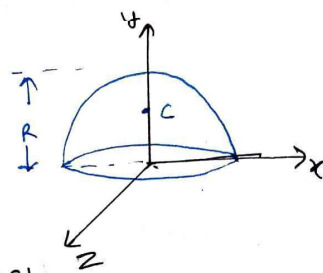
a) Cylinder:-

COV = $(\frac{L}{2}, 0, 0)$



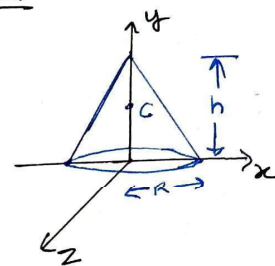
b) Hemisphere:-

COV = $(0, \frac{3R}{8}, 0)$



c) Right Circular cone:-

COV = $(0, \frac{h}{4}, 0)$



Que (1): A Square is punched out of circular lamina, the diagonal of the square being the radius of the circle as given. Find the centre of gravity of the remainder, if 'r' is the radius of the circle.

Let A be the reference point.

$$A_1 = \pi r^2$$

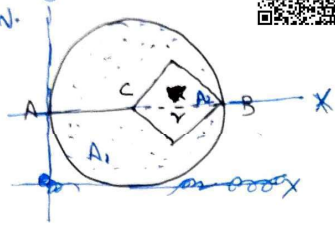
$$A_2 = 2 \left[\frac{1}{2} \cdot r \cdot \frac{r}{2} \right] = \frac{r^2}{2}$$

$$x_1 = r$$

$$x_2 = r + \frac{r}{2} = \frac{3r}{2}$$

$$A\bar{x} = A_1 x_1 - A_2 x_2 = \pi r^2 \cdot r - \frac{r^2}{2} \cdot \frac{3r}{2}$$

$$\bar{x} = \frac{\pi r^3 - \frac{3r^3}{4}}{\pi r^2 - \frac{r^2}{2}} = \frac{2\pi r^3 - 3r^3}{2\pi r^2 - r^2} = 0.905 r \text{ Ans}$$



Que (2): A semicircular area is removed from a trapezium as given. Compute the centroid of the remaining area.

Let A be the Reference Point.

$$A_1 = \frac{\pi r^2}{2} = \frac{\pi \times 20 \times 20}{2} = \pi \times 200 = 628.3$$

$$x_1 = 40 + \frac{40}{2} = 60$$

$$y_1 = \frac{4r}{3\pi} = \frac{4 \times 20}{3\pi} = 8.48 \text{ mm}$$

$$A_2 = bd = 20 \times 30 = 2400 \text{ mm}^2$$

$$x_2 = 40 \text{ mm}$$

$$y_2 = 15 \text{ mm}$$

$$A_3 = \frac{1}{2} \times 80 \times 30 = 1200 \text{ mm}^2$$

$$x_3 = \frac{2h}{3} = \frac{2 \times 20}{3} = 13.3 \text{ mm}$$

$$y_3 = 30 + \frac{1}{3} \times 30 = 40 \text{ mm}$$

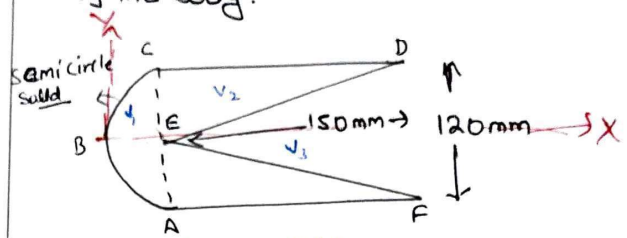
$$\bar{x} = \frac{A_2 x_2 + A_3 x_3 - A_1 x_1}{A_2 + A_3 - A_1}$$

$$\bar{x} = \frac{(2400 \times 40) + (1200 \times 13.3) - (628.3 \times 60)}{2400 + 1200 - 628.3} = 41.14 \text{ mm}$$

similarly

$$\bar{y} = 26.47 \text{ mm}$$

Que (3): A solid consist of a right circular cylinder and a hemisphere. with a cone cut out from the cylinder as given. Find the centre of gravity of the body.



Let B be a Reference Point -

$$V_1 = \frac{2}{3} \pi r^3 = \frac{2}{3} \pi (60)^3 = 4.52 \times 10^5 \text{ mm}^3, x_1 = \frac{5r}{8} = 37.5 \text{ mm}, y_1 = 0$$

$$V_2 = \pi r^2 h = \pi (60)^2 \times 150 = 1.69 \times 10^6 \text{ mm}^3, x_2 = 60 + \frac{150}{2} = 135 \text{ mm}, y_2 = 0$$

$$V_3 = \frac{1}{3} \pi r^2 h = \frac{\pi (60)^2 \times 150}{3} = 5.65 \times 10^5 \text{ mm}^3, x_3 = 60 + \frac{3 \times 150}{4} = 172.5 \text{ mm}, y_3 = 0$$

$$\bar{x} = \frac{V_1 x_1 + V_2 x_2 - V_3 x_3}{V_1 + V_2 - V_3}$$

$$= \frac{(37.5 \times 4.52 \times 10^5) + (1.69 \times 10^6 \times 135) - (5.65 \times 10^5 \times 172.5)}{4.52 \times 10^5 + 1.69 \times 10^6 - 5.65 \times 10^5}$$

$$\bar{x} = 96.6 \text{ mm}$$

$$\bar{y} = 0$$

Que (4): A solid hemisphere of 20mm radius supports a solid cone of the same base and 60mm height as given. If the upper portion of the cone is removed by a certain section, the centre of gravity lowers down by 5mm. Find the depth of section plane (h) below the apex.

Let 'A' be the Reference Point

$$V_1 = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (20)^2 \times 60 = 2513.7 \text{ mm}^3$$

$$x_1 = 0, y_1 = \frac{3}{4} \times 60 = 45 \text{ mm}$$

$$V_2 = \frac{2}{3} \pi r^3 = \frac{2}{3} \times \pi \times 20^3 = 16.75 \times 10^3 \text{ mm}^3$$

$$x_2 = 0, y_2 = 60 + \frac{3 \times 20}{8} = 67.5 \text{ mm}$$

$$\bar{y} = \frac{V_1 y_1 + V_2 y_2}{V_1 + V_2} = 54 \text{ mm}$$

Now after cutting the section $\bar{y} = 54 + 5 = 59 \text{ mm}$.

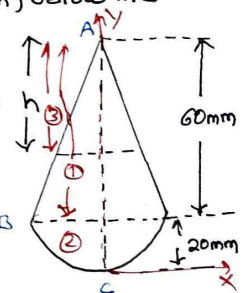
$$V_3 = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{h}{3}\right)^2 \cdot h \quad \left(\frac{20}{10} = \frac{r_3}{h} \Rightarrow r_3 = \frac{h}{3}\right)$$

$$y_3 = \frac{3h}{4}$$

$$\text{New } \bar{y} = \frac{V_1 y_1 + V_2 y_2 - V_3 y_3}{V_1 + V_2 - V_3} = 59 \text{ mm}$$

$$\bar{y} = \frac{(2513.7 \times 45) + (16.75 \times 10^3 \times 67.5) - \left(\frac{1}{3} \pi \left(\frac{h}{3}\right)^2 \cdot h \times \frac{3h}{4}\right)}{2513.7 + 16.75 \times 10^3 - \frac{1}{3} \pi \left(\frac{h}{3}\right)^2 \cdot h} = 59$$

$$h = 15.63 \text{ mm}$$



I # MOMENT OF INERTIA



Resistance towards any change (either in motion or in deformation) is termed as inertia.

The moment of force (F) about a point is termed as "first moment of force" and is equal to product of force and perpendicular distance between the point and line of action.

This moment when again multiplied by the perpendicular distance between point and line of action, this is termed as "Moment of moment of force" (MOMOF) OR Second Moment of Force (SMOF) OR moment of Inertia (MOI OR MI).

Instead of force, Area and mass can also be used.

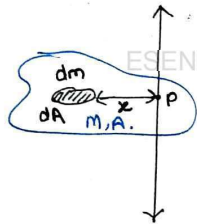
a) moment of moment of mass (MOMDM/SMOM/MOI)

$$\text{MOM/FMOM} = dm \cdot x$$

$$dIp = \text{MOMDM/SMOM} = dm \cdot x^2$$

$$Ip = \int dIp = \int dm \cdot x^2$$

- This SMOM gives the resistance towards rotation.



b) Moment of moment of Area (MOMOA/SMOA/MOI)

$$\text{MOA/FMOA} = dA \cdot x$$

$$dIp = \text{MOMOA/SMOA} = dA \cdot x^2$$

$$Ip = \int dIp = \int dA \cdot x^2$$

- This SMOA gives the resistance against deformation (Bending OR Torsion)

Written By
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Moment of Inertia of standard cases:-

[A] MOMOA/SMOA/MOI:-

(i) Rectangular Area:-

$$dIx = dAy^2$$

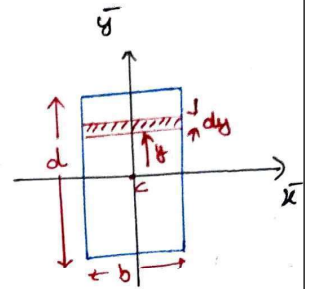
$$Ix = \int dIx = \int dAy^2$$

$$= \int_{-d/2}^{d/2} b dy \cdot y^2$$

$$= \int_{-d/2}^{d/2} by^2 dy = b \left[\frac{y^3}{3} \right]_{-d/2}^{d/2} = \frac{2b}{3} \left(\frac{d}{2} \right)^3$$

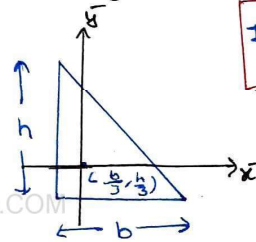
$$Ix = \frac{bd^3}{12}$$

$$\text{Similarly } Iy = \frac{db^3}{12}$$



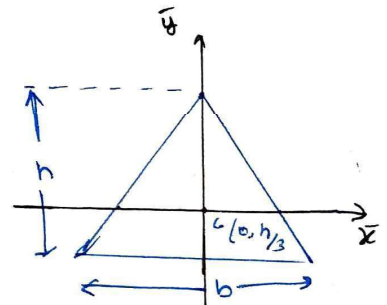
(ii) Triangular Area:-

$$Ix = \frac{bh^3}{36}, \quad Iy = \frac{hb^3}{36}$$



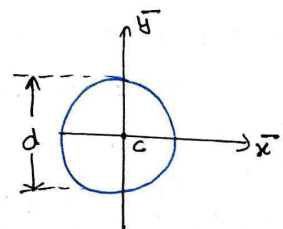
$$Ix = \frac{bh^3}{36}$$

$$Iy = \frac{hb^3}{48}$$



(iii) Circular Area:-

$$Ix = Iy = \frac{\pi d^4}{64}$$



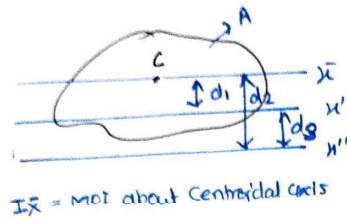
NOTE :-



A Parallel Axis THEOREM I-

If MOI about the centroidal axis is known, then the MOI about any other axis parallel to the centroidal axis can be computed using this theorem.

$$\begin{aligned} I_{x'} &= I_{\bar{x}} + Ad_1^2 \\ I_{x''} &= I_{\bar{x}} + Ad_2^2 \\ I_{x''} &\neq I_{x'} + Ad_3^2 \end{aligned}$$



Hence, this theorem is applicable when atleast one of the axis is centroidal axis i.e. It can be applied between two non-centroidal axis.

Also, MOI increases with increase in distance from centroidal axis Hence -

$$MOI_{min} = I_{\bar{x}} \quad (d=0)$$

B Perpendicular Axis THEOREM I-

If MOI is known about two mutually perpendicular centroidal axis in a plane, then the MOI about the third mutually perpendicular axis to the plane can be computed using this theorem.

$$I_z = I_{\bar{x}} + I_{\bar{y}}$$

Combined // and L axis the m.

$$I_{x''} = I_{\bar{x}} + Ad_1^2 \quad \text{--- (i)}$$

$$I_{y''} = I_{\bar{y}} + Ad_2^2 \quad \text{--- (ii)}$$

from (i) and (ii)

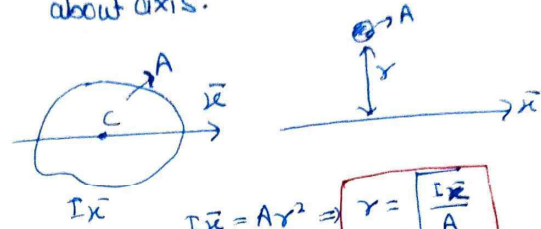
$$I_{x''} + I_{y''} = I_{\bar{x}} + I_{\bar{y}} + A(d_1^2 + d_2^2)$$

$$I_{z'} = I_z + Ar^2$$

Parallel Perpendicular axis theorem is valid for all cases of 'AMOI' and 'mmOI' but Perpendicular axis theorem is valid for all cases AMOI and for mmOI is planar body's only. (For example - it is not applicable for cylinder, sphere, cone etc)

C Radius of Gyration!-

It is the equivalent radius or distance from the centroidal axis at which whole area/mass will be placed in concentrated form such that the value of MOI would remain same about axis.



"generally it is useful for irregular body's."

$$\begin{aligned} I_{\bar{x}} &= Ar^2 \Rightarrow r = \sqrt{\frac{I_{\bar{x}}}{A}} \\ \text{OR} \\ r &= \sqrt{\frac{I_{\bar{x}}}{m}} \end{aligned}$$

Ques 5 Compute the Area moment of inertia of a semi circular section.

∴ MOI of circular section

$$I_{\bar{x}} = \frac{\pi d^4}{64}$$
 ∴ for semi-circle

$$I_{\bar{x}} = \frac{\pi d^4}{64} \times \frac{1}{2} = \frac{\pi d^4}{128}$$

$$I_x = I_{\bar{x}} + Ax^2$$

$$\frac{\pi d^4}{128} = I_{\bar{x}} + \frac{\pi d^2}{4} \times \frac{1}{2} \left(\frac{2d}{3\pi} \right)^2$$

$$I_{\bar{x}} = \frac{d^4}{145.77}$$

Ques 6 Compute the moment of inertia about the centroidal axis x-bar and y-bar of the angle section

Let 'A' point be the reference point -

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

$$x_1 = 10 \text{ mm}; y_1 = 50 \text{ mm}$$

$$A_2 = 60 \times 20 = 1200 \text{ mm}^2$$

$$x_2 = 20 + 30 = 50 \text{ mm}; y_2 = 10 \text{ mm}$$

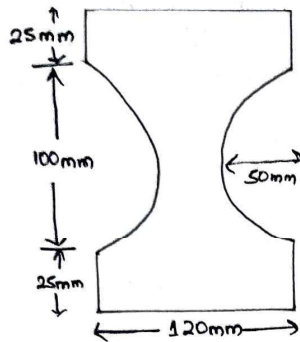
$$\bar{x} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2} = \frac{(2000 \times 10) + (1200 \times 50)}{2000 + 1200} = 25 \text{ mm}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{(2000 \times 50) + (1200 \times 10)}{2000 + 1200} = 35 \text{ mm}$$

$$\begin{aligned} I_{\bar{x}} &= I_{x_1} + A_1 r_1^2 + I_{x_2} + A_2 r_2^2 \\ I_{\bar{x}} &= \left[\frac{20 \times 100^3}{12} + (20 \times 100) (50 - 35)^2 \right] + \left[\frac{60 \times 20^3}{12} + (60 \times 20) (35 - 10)^2 \right] \end{aligned}$$

$$\begin{aligned} I_{\bar{x}} &= (2.11 \times 10^6 \text{ mm}^4) + \left[\frac{60 \times 20^3}{12} + (60 \times 20) (35 - 10)^2 \right] \\ I_{\bar{y}} &= \left[\frac{100 \times 20^3}{12} + (20 \times 100) (25 - 10)^2 \right] + \left[\frac{60^3 \times 20}{12} + (60 \times 20) (20 + 30 - 25)^2 \right] \\ I_{\bar{y}} &= 1.206 \times 10^6 \text{ mm}^4 \end{aligned}$$

Ques-7) Determine the moment of Inertia of section about horizontal and vertical axis passing through the centroid of the section



$I_{\bar{x}} = \text{MoI of Rec.} - \text{MoI of Circle}$

$$I_{\bar{x}} = \frac{bd^3}{12} - \frac{\pi d^4}{64} = \frac{120 \times 150^3}{12} - \frac{\pi (100)^4}{64} = 2.824 \times 10^7$$

$$I_{\bar{y}} = \frac{db^3}{12} - 2 \left[\frac{dc^4}{145.77} + \frac{\pi dc^2}{8} \left(\frac{b}{2} - \frac{2dc}{3\pi} \right)^2 \right]$$

$$= \frac{150 \times 120^3}{12} - 2 \left[\frac{100^4}{145.77} + \frac{\pi (100)^2}{8} \left(\frac{120}{2} - \frac{2 \times 100}{3\pi} \right)^2 \right]$$

$$I_{\bar{y}} = 8.41 \times 10^6 \text{ mm}^4$$

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Written By
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(II) Mass-moment of Inertia - mmom

(a) For a general planar body -

$$dI_{mx} = dm y^2$$

$$I_{mx} = \int dI_{mx} = \int dm y^2$$

Similarly

$$I_{my} = \int dm x^2$$

$$\text{Now, } m = \rho V = \rho A t$$

$$dm = \rho dV = \rho t dA$$

Hence -

$$I_{mx} = \int y^2 \rho t dA = \rho t \int y^2 dA = \rho t I_x$$

$$\text{Simi, } I_{mx} = \rho t I_x$$

Similarly:-

$$I_{my} = \rho t I_y$$

(b) For Rectangular Plate:-

$$I_{mx} = \rho t I_x$$

$$= \rho t \frac{bd^3}{12}$$

$$= \rho t (bd) \frac{d^2}{12}$$

$$= \rho t A \frac{d^2}{12}$$

$$= \rho V \frac{d^2}{12}$$

$$I_{mx} = \frac{M d^2}{12}$$

$$\text{Similarly } I_{my} = \frac{M b^2}{12}$$

(c) For Triangular Plate:-

$$I_{mx} = \frac{M h^2}{18}$$

$$I_{my} = \frac{M b^2}{24}$$

(d) For circular Plate:-

$$I_{mz} = \frac{m r^2}{2}$$

$$\text{also } I_{mz} = I_{mx} + I_{my}$$

and

$$I_{mx} = I_{my}$$

$$I_{mx} = I_{my} = \frac{m r^2}{4}$$

(e) For Circular Ring:-

Here $I_{mx} \neq \rho t I_x$ as no area is formed.

In this case due to presence of the entire material along the circumference.

$$\text{But } I_{mz} = m r^2$$

$$\text{also } I_{mz} = I_{mx} + I_{my} \text{ and } I_{mx} = I_{my}$$

$$I_{mx} = I_{my} = \frac{m r^2}{2}$$

(f) For Non-Planar bodies:-

(i) Sphere

$$I_{mx} = I_{my} = I_{mz} = \frac{2}{5} m r^2$$

(ii) Right-circular Cone

$$I_{mz} = \frac{3}{10} m r^2$$

$$I_{mx} = I_{my} = \frac{3}{8} m \left(\frac{r^2}{4} + h^2 \right)$$

$$I_{mx} = I_{m\bar{x}} + m \left(\frac{3h}{4} \right)^2$$

$$I_{m\bar{x}} = I_{mx} - m \left(\frac{3h}{4} \right)^2$$

(iii) Cylinder:-

$$I_{m\bar{x}} = I_{m\bar{y}} = \frac{m}{12} (3r^2 + l^2)$$

Here, (a) If $r \gg l$
Cylinder $\rightarrow$ disc

$$I_{m\bar{x}} = I_{m\bar{y}} = \frac{m r^2}{4}$$

(b) If $r \ll l$

Cylinder $\rightarrow$ Rod/Bar

$$I_{m\bar{x}} = I_{m\bar{y}} = \frac{m l^2}{12}$$

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Hence -

$$I_{mx} = \int y^2 \rho t dA = \rho t \int y^2 dA = \rho t I_x$$

$$\text{Simi } I_{mx} = \rho t I_x$$

Similarly:-

$$I_{my} = \rho t I_y$$

(b) For Rectangular Plate:-

$$\begin{aligned} I_{mx} &= \rho t I_x \\ &= \rho t \frac{bd^3}{12} \\ &= \rho t (bd) \frac{d^2}{12} \\ &= \rho t A d^2 \frac{1}{12} \\ &= \rho V d^2 \frac{1}{12} \end{aligned}$$

$$I_{mx} = \frac{M d^2}{12}$$

$$\text{Similarly } I_{my} = \frac{M b^2}{12}$$

(c) For Triangular Plate:-

$$I_{mx} = \frac{M h^2}{18}$$

$$I_{my} = \frac{M b^2}{24}$$

(d) For circular Plate:-

$$I_{mz} = \frac{m r^2}{2}$$

$$\text{also } I_{mz} = I_{mx} + I_{my} \text{ and } I_{mx} = I_{my}$$

$$I_{mx} = I_{my} = \frac{m r^2}{4}$$

(e) For Circular Ring:-

Here $I_{mx} \neq \rho t I_x$ as no area is formed.

In this case due to presence of the entire material along the circumference.

$$\text{But } I_{mz} = m r^2$$

$$\text{also } I_{mz} = I_{mx} + I_{my} \text{ and } I_{mx} = I_{my}$$

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(i) Sphere

$$I_{mx} = I_{my} = I_{mz} = \frac{2}{5} m r^2$$

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$$I_{mz} = \frac{3}{10} m r^2$$

$$I_{mx} = I_{my} = \frac{3}{8} m \left(\frac{r^2}{4} + h^2 \right)$$

$$I_{mx} = I_{m\bar{x}} + m \left(\frac{3h}{4} \right)^2$$

$$I_{m\bar{x}} = I_{mx} - m \left(\frac{3h}{4} \right)^2$$

(iii) Cylinder:-

$$\begin{aligned} I_{m\bar{x}} &= I_{m\bar{y}} \\ &= \frac{m}{12} (3r^2 + l^2) \end{aligned}$$

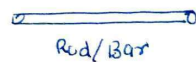
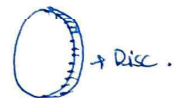
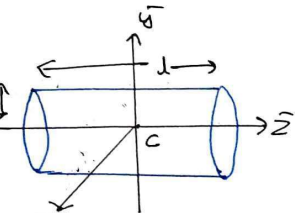
Here, (a) If $r \gg l$
Cylinder $\rightarrow$ disc

$$I_{m\bar{x}} = I_{m\bar{y}} = \frac{m r^2}{4}$$

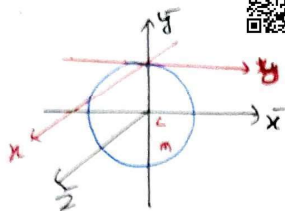
(b) If $r \ll l$

Cylinder $\rightarrow$ Rod/Bar

$$I_{m\bar{x}} = I_{m\bar{y}} = \frac{m l^2}{12}$$



Ques ① For given disc of mass "m" and radius "r", find its M.M.O.I about x and y axis.

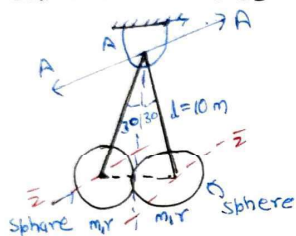


M.M.O.I about Noncentrroidal axis = M.M.O.I about Centroidal axis + mr^2

$$I_{mx} = I_{mz} + mr^2 = \frac{mr^2}{2} + mr^2 = \frac{3}{2}mr^2$$

$$I_{my} = I_{mx} + mr^2 = \frac{mr^2}{4} + mr^2 = \frac{5}{4}mr^2$$

Ques ② Find the total M.M.O.I of the assembly about hing "A", both bars are identical and massless.



Solution $\Rightarrow$

$$I_A = (I_z + ml^2)2 = \left(\frac{2}{5}mr^2 + ml^2\right)2$$

$$= 2\left[\frac{2}{5}m(1.5 \sin 30^\circ)^2 + ml^2\right] = 22.20M$$

VIRTUAL WORK! -

- If force 'F' on a body displaces it through distance 's' then work done = $F \cdot s$

- In some cases, the body does not move in the direction of force in such case work done = $F \cos \theta \cdot s$

θ = Angle between line of action of force and direction of displacement.

- If $0 \leq \theta < 90^\circ \Rightarrow$ Work done (+ve)

$\theta = 90^\circ \Rightarrow$ Work done (0)

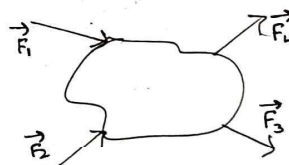
$90^\circ < \theta < 180^\circ \Rightarrow$ Work done (-ve)



CONCEPT OF VIRTUAL WORK! -

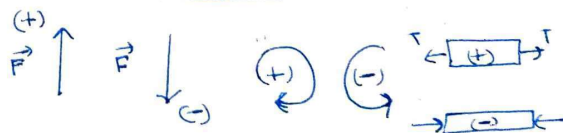
- If the body is in equilibrium under the action of system of forces, then work done by these force is zero.

- The body under equilibrium may be assumed to undergo an infinite small imaginary displacement (termed as virtual displacement), so the virtual work done by all these force would be zero as body is in equilibrium.



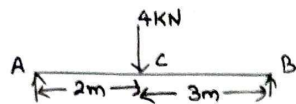
(NOTE)

Sign Convention! -

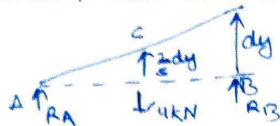


Written By
A. HANNAN

Que ① For the given beam, find the reactions, using the principle of virtual work



Let A be the reference point, and give virtual displacement about point 'B' i.e. dy .



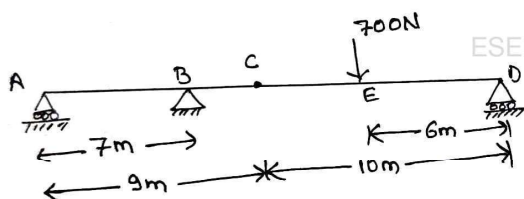
$$R_A \times 0 + R_B dy + (-4) \frac{2}{5} dy = 0$$

$$R_B = 1.6 \text{ kN}$$

$$R_A = 4 - 1.6 = 2.4 \text{ kN}$$

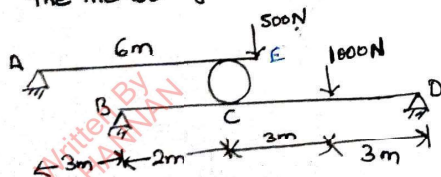
Que ② Two beams AC and CD of length 9m & 10m respectively are hinged at C. These are supported on rollers at the left and right ends, A hinged support is provided at B, 7m from A.

Using the principle of virtual work, determine the force transmitted by the hinge 'C' and the reaction at the support B, when a load of 700N at a point 6m from D

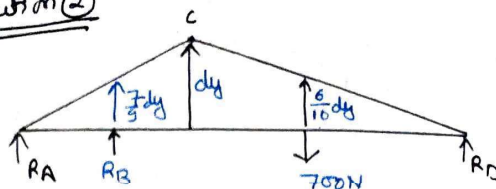


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Que ③ Two beams AE and BD are supported on rollers at B and C. Compute the reaction's at the roller B and C using the method of virtual work.



Solution ②

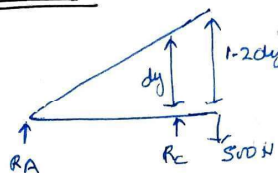


$$R_A \times 0 + R_B \times \frac{7}{5} dy + (-700) \frac{6}{10} dy + R_D \times 0 = 0$$

$$\frac{7}{5} R_B = \frac{10}{70} \times 6$$

$$R_B = \frac{60 \times 9}{8} = \frac{720}{8} = 90 \text{ N}$$

Solution ③

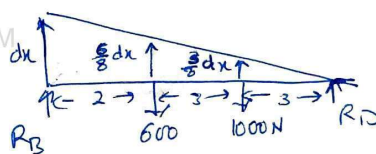


For Beam - AB

$$R_A \times 0 + R_B \times dy + (500) 1.2 dy = 0$$

$$R_C = 1.2 \times 500 = 600 \text{ N}$$

For Beam BD

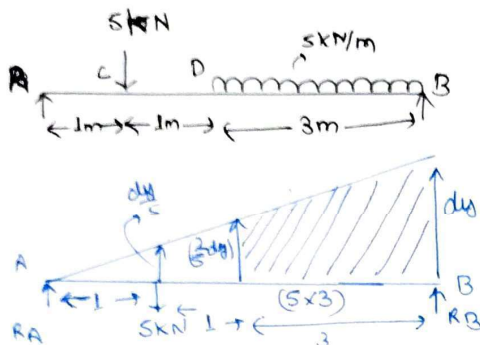


$$R_B \cdot dx + (-600) \frac{3}{8} dx + (-1000) \frac{3}{8} dx + R_D \times 0 = 0$$

$$R_B = \frac{6600}{8} = 825 \text{ N}$$

UN-17

Ques 4 A simply supported beam AB of span 5m is loaded as given using the principle of virtual work, find the reaction at A and B.



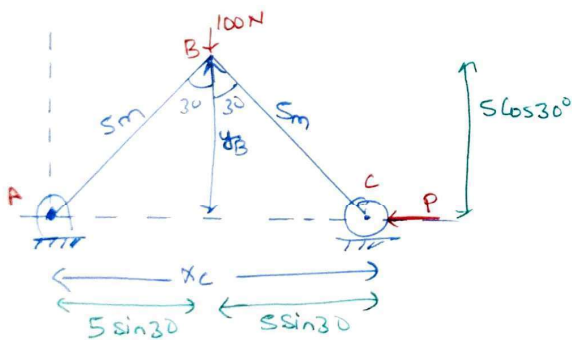
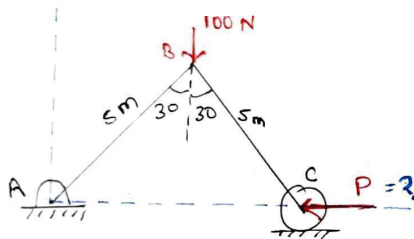
$$R_A \cdot 0 - 5 \times \frac{dy}{5} - (5 \times 3) \left[\frac{1}{2} \left(\frac{2}{5} dy + dy \right) \right] + R_B \cdot dy = 0$$

$$R_B = 11.5 \text{ kN}$$

$$R_A = \text{Total load} - R_B$$

$$R_A = 5 + (5 \times 3) - 11.5 = 9.5 \text{ kN}$$

Ques 5 Find "P" for the equilibrium of the system at given position. Both the bars are identical and massless.



Solution

$$-100 dy_B - P dx_C = 0$$

$$-100 d(5 \cos 30) - P d(10 \sin 30) = 0$$

differentiate differentiate

$$-100 (-5 \sin 30) - P (10 \cos 30) = 0$$

$$+ 500 \times \frac{1}{2} - 10P \frac{\sqrt{3}}{2} = 0$$

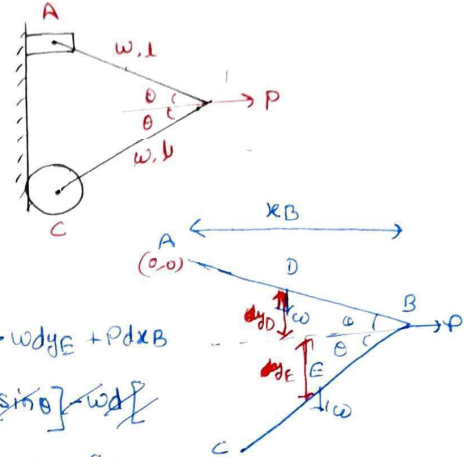
$$P = 28.86 \text{ N}$$

UN-18 - Fmly

UN-19



Ques 6 Find the value of force "P" to keep system in equilibrium at the given position. Both bars are identical with weight "w" and length "l". Use principle of virtual work.



$$-w dy_D - w dy_E + P dx_B$$

$$-w d \left[\frac{l}{2} \sin \theta \right] - w d \left[\frac{l}{2} \cos \theta \right]$$

$$-w \left[-\frac{l}{2} \cos \theta d\theta \right] - w \left[-\frac{l}{2} \sin \theta d\theta \right]$$

$$+ P [-l \sin \theta d\theta] = 0$$

$$[P = 2w \cos \theta]$$

$$y_D = \frac{l}{2} \sin \theta$$

$$y_E = \frac{l}{2} \cos \theta$$

$$x_B = l \cos \theta$$

$$dy_D = \frac{l}{2} \cos \theta d\theta$$

$$dy_E = -\frac{l}{2} \sin \theta d\theta$$

$$dx_B = -l \sin \theta d\theta$$

Ques 7

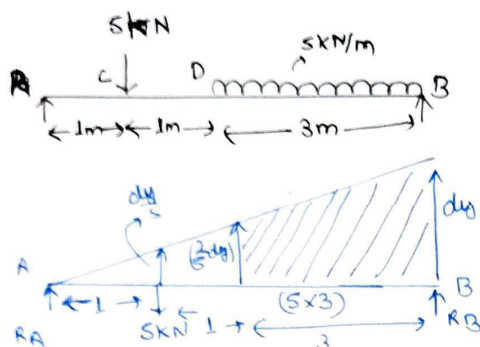
Two identical weightless bars are holding a cylinder in equilibrium at given position. All surfaces are smooth and cylinder has $w = 1000 \text{ N}$ and $r = 10 \text{ m}$. Then find required force "F" on both the bars for equilibrium of system.



P.T.O.

UN-17

Ques 4 A simply supported beam AB of span 5m is loaded as given using the principle of virtual work, find the reaction at A and B.



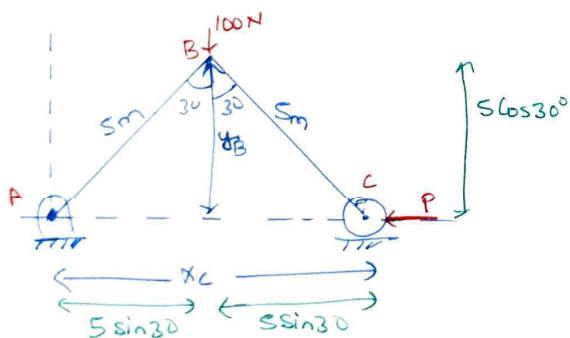
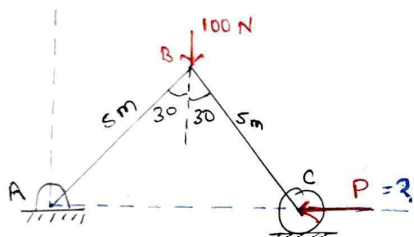
$$\delta U = 0 \Rightarrow 5 \times \frac{dy}{5} - (3 \times 3) \left[\frac{1}{2} \left(\frac{2}{3} dy + dy \right) \right] + R_B \cdot dy = 0$$

$$R_B = 11.5 \text{ kN}$$

$$R_A = \text{Total Load} - R_B$$

$$R_A = 5 + (3 \times 3) - 11.5 = 9.5 \text{ kN}$$

Ques 5 Find "P" for the equilibrium of the system at given position. Both the bars are identical and massless.



Solution

$$-100 dy_B - P dx_C = 0 \Rightarrow$$

$$-100 d(5 \cos 30) - P d(10 \sin 30) = 0$$

differentiate differentiate

$$-100 (5 \sin 30) - P (10 \cos 30) = 0$$

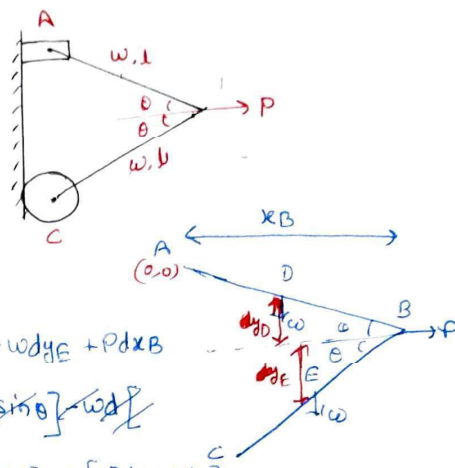
$$-500 \times 1 - 10P \frac{\sqrt{3}}{2} = 0$$

$$P = 28.86 \text{ N}$$

UN-18 - FmHj

UN-19

Ques 6 Find the value of force "P" to keep system in equilibrium at the given position. Both bars are identical with weight "w" and length "l". use principle of Virtual work.



$$\delta U = 0 \Rightarrow -w dy_D - w dy_E + P dx_B = 0$$

$$-w \left[\frac{l}{2} \sin \theta \right] - w \left[\frac{l}{2} \cos \theta \right]$$

$$-w \left[-\frac{l}{2} \cos \theta d\theta \right] - w \left[-\frac{l}{2} \sin \theta d\theta \right]$$

$$+ P [-l \sin \theta d\theta] = 0$$

$$[P = 2w \cot \theta]$$

$$\theta = \frac{1}{2} \sin \theta$$

$$dy_D = \frac{l}{2} \sin \theta$$

$$dy_E = \frac{l}{2} \cos \theta$$

$$dx_B = l \cos \theta$$

$$dy_D = -\frac{l}{2} \cos \theta d\theta$$

$$dy_E = -\frac{3l}{2} \cos \theta d\theta$$

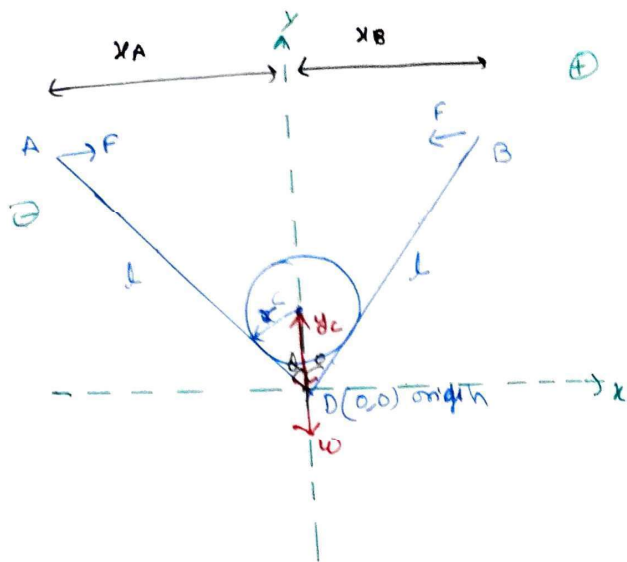
$$dx_B = -l \sin \theta d\theta$$

Ques 7

Two identical weightless bars are holding a cylinder in equilibrium at given position. All surfaces are smooth and cylinder has $w = 1000 \text{ N}$ and $r = 10 \text{ m}$. Then find required force 'F' on both the bars for equilibrium of system.



P.T.O.

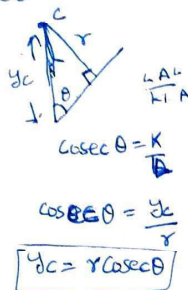


$$x_A = -L \sin \theta, \quad dx_A = -L \cos \theta \cdot d\theta$$

$$x_B = L \sin \theta, \quad dx_B = L \cos \theta \cdot d\theta$$

$$y_C = L \sec \theta$$

$$dy_C = -L \sec \theta \cdot \tan \theta \cdot d\theta$$



$$\Rightarrow -W \cdot dy_C + F \cdot dx_A - F \cdot dx_B = 0$$

$$\Rightarrow 1000 [-L \sec \theta \cdot \tan \theta \cdot d\theta] + F [-L \cos \theta \cdot d\theta] - F [L \cos \theta \cdot d\theta] = 0$$

$$\Rightarrow 1000 [-10 \sec \theta \cdot \tan \theta \cdot d\theta] + F [-L \cos \theta \cdot d\theta] - F [L \cos \theta \cdot d\theta] = 0$$

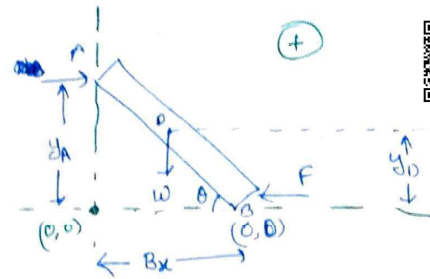
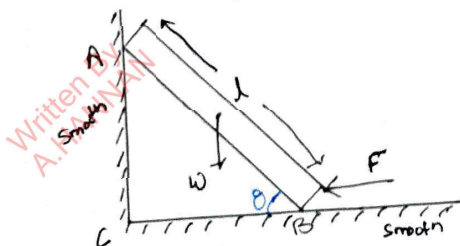
$$\Rightarrow 10000 \sec \theta \cdot \tan \theta \cdot d\theta - 2FL \cos \theta \cdot d\theta = 0$$

$$\Rightarrow 10000 \sec \theta \cdot \tan \theta \cdot d\theta = 2FL \cos \theta \cdot d\theta$$

$$\Rightarrow 10000 \frac{1}{\sin \theta} \times \frac{\cos \theta}{\sin \theta} d\theta = 2FL \cos \theta \cdot d\theta$$

$$\Rightarrow \frac{10000 \cdot \cos^2 \theta}{2L} = F \quad \text{Ans}$$

Ques 3 A bar of weight $W = 1000\text{N}$, $L = 100\text{m}$, placed on a smooth floor and vertical wall as given. Find the required force "F" to keep bar is equilibrium at given position. Take $\theta = 45^\circ$.



$$y_D = \frac{L}{2} \sin \theta; \quad dy_D = \frac{L}{2} \cos \theta \cdot d\theta$$

$$B_x = \frac{L}{2} \cos \theta; \quad dx_B = -L \sin \theta \cdot d\theta$$

$$-W \cdot dy_D - F \cdot dx_B = 0$$

$$-W \cdot \frac{L}{2} \cos \theta \cdot d\theta - F (-L \sin \theta \cdot d\theta) = 0$$

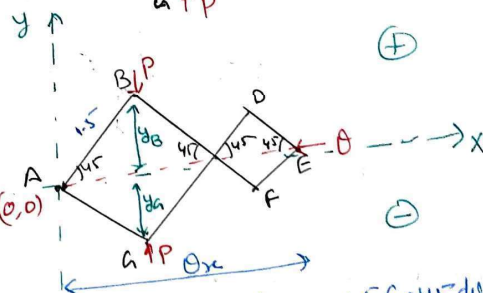
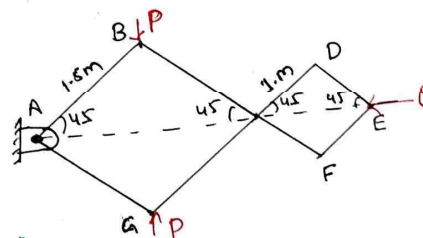
$$F \cdot L \sin \theta \cdot d\theta = \frac{WL}{2} \cos \theta$$

$$F = \frac{W \cos \theta}{2 \sin \theta} = \frac{W \cot 45^\circ}{2} = \frac{W}{2}$$

Ques 4 If all members are weightless and joints are frictionless, then find relation b/w 'P' and 'Q' for equilibrium of system at given position.

$$a) P = 6Q \quad b) 0.6\theta = P$$

$$c) Q = 6P \quad d) \theta = 0.6P$$



$$y_B = 1.5 \sin 45^\circ; \quad dy_B = 1.5 \cos 45^\circ \cdot d\theta$$

$$y_G = 1.5 \sin 45^\circ; \quad dy_G = 1.5 \cos 45^\circ \cdot d\theta$$

$$B_x = 1.5 \cos 45^\circ + 1.5 \cos 45^\circ + 1.5 \cos 45^\circ + 1.5 \cos 45^\circ$$

$$dx_B = -1.5 \sin 45^\circ - 1.5 \sin 45^\circ - 1.5 \sin 45^\circ - 1.5 \sin 45^\circ$$

$$dx_B = -5 \sin 45^\circ \cdot d\theta$$

$$\Rightarrow -P \cdot dy_B + P \cdot dy_G - Q \cdot dx_B = 0$$

$$= -P \cdot 1.5 \cos 45^\circ + P \cdot 1.5 \cos 45^\circ - Q \cdot (-5 \sin 45^\circ) = 0$$

$$= -1.5P \cos 45^\circ + 5Q \sin 45^\circ = 0$$

$$= -1.5P \cos 45^\circ + 5Q \sin 45^\circ = 0$$

$$5Q = 1.5P \quad \Rightarrow Q = \frac{3}{10}P$$

By Jaspal Sir

TRANSLATION MOTION



A Body or particle is said to be in motion if its position is continuously changing w.r.t. some fixed point (origin).

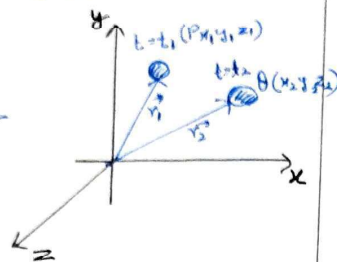
$P(x, y, z)$ and $x, y, z = f(t)$

If this translation motion is in -

a) 1D: $\vec{r} = x\hat{i}, y\hat{j}, z\hat{k}$

b) 2D: $\vec{r} = x\hat{i} + y\hat{j}, x\hat{i} + z\hat{k}$
 $y\hat{j} + z\hat{k}$

c) 3D: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$



- Distance (s):- It is the total length of path covered by the body b/w two concerned points

- Displacement ($\vec{r}$):- It is the shortest length b/w any two concerned points.

- Speed (v):- It is rate of change of distance.

- Velocity ($\vec{v}$):- It is the rate of change of displacement.

- Average Velocity ($\vec{v}_{avg}$):- $\frac{\text{Total displacement}}{\text{Total time}}$

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

- Instantaneous Velocity ($\vec{v}_{inst}$) = $\lim_{t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$

- Acceleration:- It is rate of change of velocity

- Average acceleration ($\vec{a}_{avg}$)

$$= \frac{\text{Total change in velocity}}{\text{Total Time}}$$

$$= \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

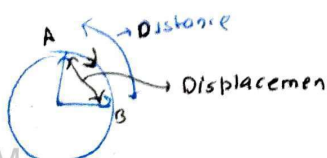
- Instantaneous acceleration:- $(\vec{a}_{avg}) \rightarrow (\vec{a}_{inst})$

$$= \lim_{t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

$$\frac{OR}{|\vec{a}_{inst}|} = \left| \frac{d\vec{v}}{dt} \right| = \frac{dv}{dt} \cdot \frac{ds}{ds} = v \frac{dv}{ds}$$

- JERK:- It is defined rate of change of acceleration.

$$\vec{J} = \frac{d\vec{a}}{dt}$$



EQUATION OF MOTION:-

$$\begin{aligned} \text{a) } v &= u + at \\ \text{b) } v^2 &= u^2 + 2as \\ \text{c) } s &= ut + \frac{1}{2}at^2 \\ \text{d) } s &= \frac{1}{2}(u+v)t \end{aligned} \quad \left\{ \begin{array}{l} u = \text{initial velocity} \\ v = \text{Final velocity} \\ s = \text{displacement} \\ t = \text{time} \\ a = \text{acceleration / retardation} \end{array} \right.$$

Ques 1:- If a particle is in rectilinear motion given by $x = t^3 - 2t^2 + 10t + 6$ then find -

a) position, velocity and acceleration at $t = 4$ sec.

b) v_{avg} and a_{avg} for $t = 3$ sec to $t = 4$ sec.

c) v_{inst} and a_{inst} for $t = 5$ sec.

① (i) Position:-

$$x = t^3 - 2t^2 + 10t + 6$$

$$\vec{r} = x\hat{i} = (t^3 - 2t^2 + 10t + 6)\hat{i}$$

at $t = 4$ sec -

$$\vec{r} = (4^3 - 2(4)^2 + (10 \times 4) + 6)\hat{i} = 78\hat{i}$$

(ii) Velocity:-

$$\vec{v} = \frac{d\vec{r}}{dt} = (3t^2 - 4t + 10)\hat{i}$$

$$\text{at } t = 4 \text{ sec, } \vec{v} = [3(4)^2 - 4(4) + 10]\hat{i} = 42\hat{i}$$

(iii) acceleration:-

$$\vec{a} = \frac{d\vec{v}}{dt} = 6t - 4 \quad \text{at } t = 4 \text{ sec} \\ \vec{a} = (6 \times 4 - 4)\hat{i} = 20\hat{i}$$

$$\text{② } \vec{v}_{avg} = \frac{\text{Total Displacement}}{\text{Total Time}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

$$= \frac{[4^3 - 2(4)^2 + (10 \times 4) + 6]\hat{i} - [3^3 - 2(3)^2 + (10 \times 3) + 6]\hat{i}}{2 - 1}$$

$$= 78\hat{i} - 45\hat{i} = 33\hat{i}$$

$$\text{③ } \vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

$$= \frac{[3(4)^2 - 4(4) + 10]\hat{i} - [3(3)^2 - 4(3) + 10]\hat{i}}{2 - 1} = 42\hat{i} - 25\hat{i} = 17\hat{i}$$

$$\text{④ } \vec{v}_{inst} = \frac{d\vec{r}}{dt} = 3t^2 - 4t + 10$$

$$\text{at } t = 5 \quad = (3 \times 5^2 - 4 \times 5 + 10)\hat{i} = 65\hat{i}$$

$$\vec{v}_{avg} = \frac{d\vec{v}}{dt} = (6t - 4)\hat{i} = (6 \times 5 - 4)\hat{i} = 26\hat{i}$$

Ques 2 The equation of acceleration of a particle is given as $a = kt$ where 'k' is constant and 't' is time (sec). Find the equation of motion, if particle is in rectilinear motion and it is given that at $t = 1 \text{ sec}$, $v = 2 \text{ m/sec}$, $t = 2 \text{ sec}$, $v = 3 \text{ m/sec}$ and $x = 0 \text{ m}$

$$a = \frac{dv}{dt} = kt$$

$$\Rightarrow dv = kt dt$$

$$\Rightarrow \int dv = \int kt dt$$

$$\Rightarrow v = k \frac{t^2}{2} + C_1$$

at $t = 2 \text{ sec}$, $v = 2 \text{ m/sec}$

$$2 = k \frac{(2)^2}{2} + C_1 \Rightarrow 2 = 2k + C_1 \quad \text{--- (i)}$$

at $t = 2 \text{ sec}$, $v = 3 \text{ m/sec}$

$$3 = k \frac{(2)^2}{2} + C_1 \Rightarrow 6 = 4k + 2C_1 \quad \text{--- (ii)}$$

eq (ii) - (i) $\Rightarrow 4 = 2k \Rightarrow k = \frac{2}{3}$

from eq (i) $\Rightarrow 2 = 2k + C_1$

$$4 - \frac{2}{3} = 2C_1 \Rightarrow \frac{10}{3} = 2C_1 \Rightarrow C_1 = \frac{5}{3}$$

Equation of motion

$$v = \frac{2}{3} \frac{t^2}{2} + \frac{5}{3} = \frac{t^2}{3} + \frac{5}{3}$$

$$v = \frac{dx}{dt} = \frac{t^2}{3} + \frac{5}{3} + C_2$$

at $t = 3$, $x = 0$

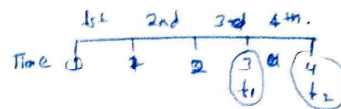
$$C_2 = -8$$

Equation of motion (2)

$$= \frac{t^3}{3 \times 3} + \frac{5t}{3} + C_2$$

$$= \frac{t^3}{9} + \frac{5t}{3} + (-8)$$

Ques 3 :- If $s = \frac{t^3}{6} - 18t$, where t is time in (sec), Find the average acceleration during 4th sec.



$$a_{avg} = \frac{v_2 - v_1}{t_2 - t_1}$$

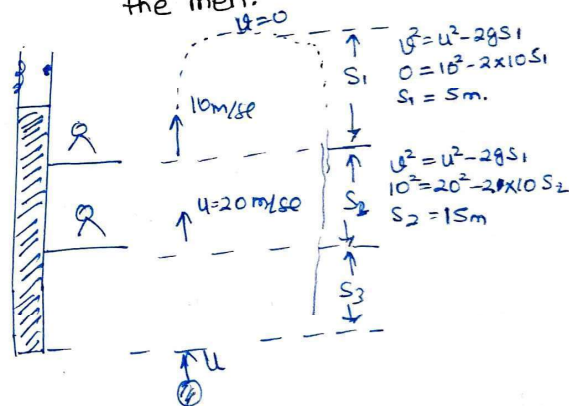
$$v_1(3) = \frac{3t^2}{6} - 18 = \frac{3^2}{2} - 18 = \frac{9 - 36}{2} = -13.5$$

$$v_2(4) = \frac{4^2}{2} - 18 = \frac{16 - 36}{2} = -10 \text{ m/sec}$$

$$a_{avg} = \frac{-10 - (-13.5)}{1} = 3.5 \text{ m/sec}^2$$

Ques 4 :- A ball is thrown vertically upward with velocity 'u' from the ground. Two men standing on a building at different floors watch the ball passing by them with 20 m/sec and 10 m/sec respectively. Find:-

- How high does the ball rise beyond the second man.
- The difference of height b/w both the men.



$$a) = S_1 = 5 \text{ m}$$

$$b) = 15 \text{ m}$$

Ques 5 A stone is thrown upward, it reaches back to the ground after 7.5 sec. Find the maximum height attained by the stone.

Also compute velocity after 5 sec.

$v=0$ at $t=3.75$
 $t=0$ to $t=7.5$
 $v = u - gt$
 $0 = u - 10 \times 3.75 = 0$
 $u = 37.5 \text{ m/sec.}$
 $v^2 = u^2 - 2gs$
 $0 = 37.5^2 - 2 \times 10 \times s$
 $s = \frac{37.5^2}{20} = 70.31 \text{ m.}$
 Velocity after 5 sec.
 $v = u - gt$
 $= 37.5 - 10 \times 5 = -12.5 \text{ m/sec.}$
 (-) sign signifies toward downward direction

2D motion

Ques 6 A particle is moving in a plane as per equation of motion given as -
 $x = 4 + 6t$, $y = 1.5 + 0.5t^3$, where x and y are in (m) and 't' is time in (seconds) Compute -

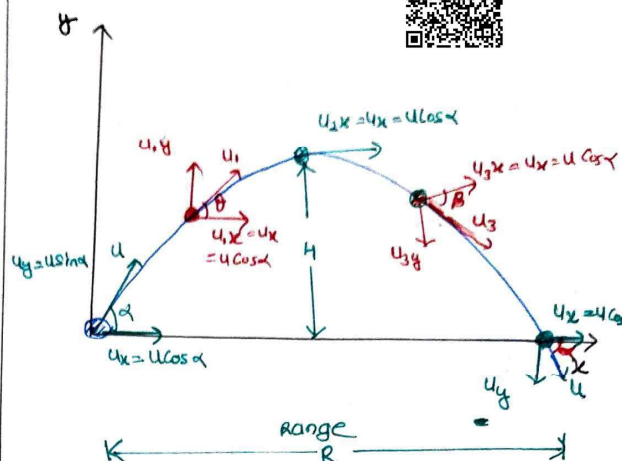
- Position, velocity and acceleration at $t = 3 \text{ sec}$
- Also find their directions.

i) $\vec{r} = x\hat{i} + y\hat{j} = (4 + 6t)\hat{i} + (1.5 + 0.5t^3)\hat{j}$
 $t = 3 \text{ sec}$
 $\vec{r} = [4 + (6 \times 3)]\hat{i} + [1.5 + 0.5(3)^3]\hat{j}$
 $= 22\hat{i} + 15\hat{j}$
 $|\vec{r}| = \sqrt{22^2 + 15^2} = 26.63 \text{ m}$; $\alpha = \tan^{-1}(\frac{y}{x})$
 $\alpha = \tan^{-1}(\frac{15}{22}) = 34.28^\circ$

ii) $\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = 6\hat{i} + 1.5t^2\hat{j}$
 $t = 3 \text{ sec.}$
 $\vec{v} = 6\hat{i} + 1.5(3)^2\hat{j}$
 $= 6\hat{i} + 13.5\hat{j}$
 $|\vec{v}| = \sqrt{6^2 + 13.5^2} = 14.77 \text{ m/sec}$
 $\beta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{13.5}{6})$
 $\beta = 66.04^\circ$

iii) $\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = (a\hat{i} + 3t\hat{j})$
 $t = 3 \text{ sec}$
 $\vec{a} = 9\hat{j}$
 $|\vec{a}| = 9 \text{ m/sec}^2$

PROJECTILE MOTION:-



- when body/particle is thrown upward at some angle with ground it moves on a curved parabolic path under gravity termed as "Projectile motion"

a) motion along y-axis:-

from A to B, $v = u + at$
 $0 = u \sin \alpha - gt$
 $t = \frac{u \sin \alpha}{g}$

Hence,

Total time of flight $T = 2t = \frac{2u \sin \alpha}{g}$

Now, $v^2 = u^2 + 2as$

$0 = (u \sin \alpha)^2 - 2gH$

Height of Projectile

$H = \frac{u^2 \sin^2 \alpha}{2g}$

For maximum height, $\sin^2 \alpha = 1$

$\Rightarrow \sin \alpha = 1$

$\Rightarrow \alpha = 90^\circ$

$H_{\max} = \frac{u^2}{2g}$

b) motion along x-axis:-

Since $a_x = 0 \Rightarrow V_x = \text{const} = u \cos \alpha$

$$x = u_x t$$

When, $t = T$, $x = \text{Range of Projectile (R)}$
 $= u_x T$

$$R = u \cos \alpha \cdot \frac{2u \sin \alpha}{g}$$

$$\left\{ \begin{array}{l} u = \frac{R}{T} \\ T = \frac{R}{u} \end{array} \right.$$

$$R = \frac{u^2 \sin 2\alpha}{g}$$

For max. range, $\sin 2\alpha = 1 \Rightarrow 2\alpha = 90^\circ$
 $\alpha = 45^\circ$

$$R_{\max} = \frac{u^2}{g}$$

Now, Equation of motion i.e. $y = f(x)$:-

$$s = ut + \frac{1}{2}at^2$$

along y-axis, $h = u_y t - \frac{g}{2}t^2$

along x-axis, $x = u_x t \Rightarrow t = \frac{x}{u_x} = \frac{x}{u \cos \alpha}$

$$h = u \sin \alpha \cdot \left(\frac{x}{u \cos \alpha} \right) - \frac{g}{2} \left(\frac{x}{u \cos \alpha} \right)^2$$

$$h = x \tan \alpha - \frac{g}{2 \cos^2 \alpha} x^2 \quad \rightarrow \text{Eqn of motion}$$

$$h = x \tan \alpha - \frac{g}{32} \left(\frac{x}{\cos \alpha} \right)^2$$

Que 1 A Projectile is fired from a point at height "h", with a horizontal velocity of 6m/sec. If it has to hit an object at 12m, then find required height "h"

$\therefore a_x = \text{acceleration} = 0$
 $u_x = \text{velocity} = \text{constant}$

$$u_x = \frac{\text{Distt (x)}}{\text{Time (t)}}$$

$$x = t = \frac{x}{u_x} = \frac{12}{6} = 2 \text{ sec.}$$

$$y = u_y t + \frac{1}{2} g t^2 = \left| 0 - \frac{1}{2} \times 10 \times 2^2 \right| = 20 \text{ m. Ans}$$



Que 2 A ball thrown from the ground with $u = 40 \text{ m/sec}$ at an angle $\alpha = 30^\circ$ with the horizontal, then find -

a) velocity of ball at 0.5 and 2.5 sec.

b) Total flight time, Height and maximum Range of projectile.

Q1: $u_x = u \cos \alpha = 40 \cos 30 = 34.64 \text{ m/sec.}$

time to reach, $h_{\max} = \frac{u \sin \alpha}{g} = \frac{40 \sin 30}{9.81} = 2.038 \text{ sec.}$

$\therefore t = 0.5 \text{ sec} < 2.038 \text{ sec} \Rightarrow \text{Retardation m.}$

$$u_{1y} = u \sin \alpha - g t = 40 \sin 30 - 9.81 \times 0.5 = 15.095 \text{ m/sec}$$

$$V_1 = \sqrt{u_{1x}^2 + u_{1y}^2} = \sqrt{34.64^2 + 15.096^2} = 37.78 \text{ m/sec.}$$

Q2: $u_x = u \cos \alpha = 40 \cos 30 = 34.64 \text{ m/sec.}$

$\therefore t = 2.5 \text{ sec} > 2.038 \text{ sec} \Rightarrow \text{Acceleration m.}$

*** $u_{2y} = u \sin \alpha + g t = 40 \sin 30 + 9.81 (2.5 - 2.038)$

$$u_{2y} = 4.53 \text{ m/sec.}$$

$$u_2 = \sqrt{u_x^2 + u_{2y}^2} = \sqrt{34.64^2 + 4.53^2} = 34.93 \text{ m/sec.}$$

6) $T = \frac{2u \sin \alpha}{g} = \frac{2 \times 40 \times \sin 30}{9.81} = 4.077 \text{ sec.}$

↓
Total Time

Height

$$\text{Height (H)} = \frac{u^2 \sin^2 \alpha}{2g} = \frac{40^2 \sin^2 30}{2 \times 9.81} = 20.38 \text{ m.}$$

$$\text{max. Rang (R)} = \frac{u^2}{g} = \frac{40^2}{9.81} = 163.09 \text{ m.}$$

7) Find the condition for which "Range" is equal to "height" of projectile.

$$\text{Range} = \text{Height}$$

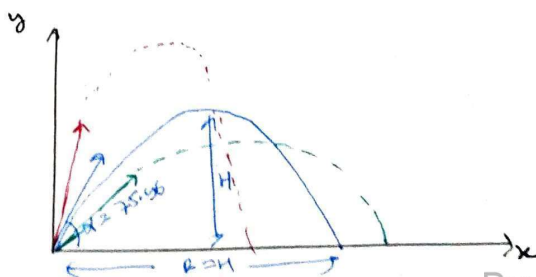
$$\frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin^2 \alpha}{2g}$$

$$2 \sin \alpha \cdot \cos \alpha = \frac{\sin \alpha \cdot \sin \alpha}{2}$$

$$4 = \frac{\sin \alpha}{\cos \alpha}$$

$$\tan \alpha = 4$$

$$\alpha = 75.96^\circ$$

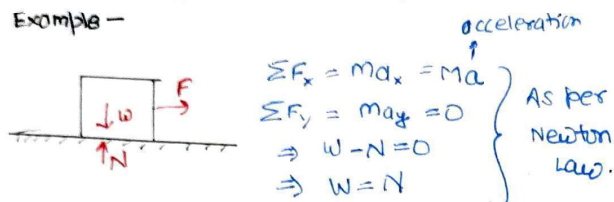




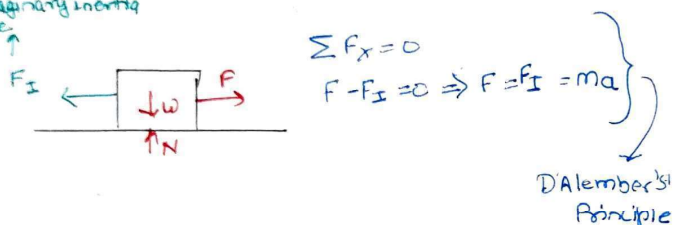
D'Alembert's Principle:-

If many forces acting on the body and it is motion with acceleration, then the body is said to be in Dynamic Equilibrium under the application of all the forces and an IMAGINARY INERTIA FORCE acting at the centre of mass of the body and in opposite direction of net acceleration.

Example -

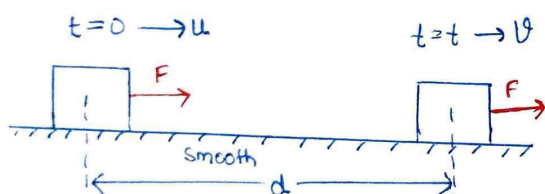


Imaginary inertia force



Work Energy Principle:-

According to this principle work and energy are interconvertible or work is the form of energy.



$$\begin{aligned}dw &= F \cdot dx \\ &= ma \cdot dx \\ &= m \cdot v \cdot \frac{dv}{dx} \cdot dx\end{aligned}$$

$$dw = m \cdot v \cdot dv$$

$$w = \int dw = \int m \cdot v \cdot dv = m \cdot \left[\frac{v^2}{2} \right]_u^v$$

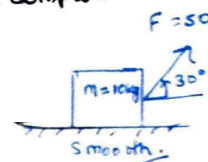
$$w = \frac{m \cdot v^2}{2} - \frac{m \cdot u^2}{2}$$

$$w = \frac{1}{2} m v^2 - \frac{1}{2} m u^2 = \Delta KE$$

ΔKE = Change in Kinetic Energy.

Ques:- If a block initially at rest is subjected to a force of 50 N is applied as shown, then compute -

- Acceleration
- Velocity after 15 sec
- Distance travelled 15 sec
- Work done in 15 sec.



$$\begin{aligned}\textcircled{1} \Sigma F_x &= ma_x \Rightarrow 50 \cos 30^\circ = 10 a_x \\ \Rightarrow a_x &= 4.33 \text{ m/sec}^2\end{aligned}$$

$$\textcircled{2} v = u + at = 0 + 4.33 \times 15 = 64.95 \text{ m/sec}$$

$$\begin{aligned}\textcircled{3} s &= ut + \frac{1}{2} at^2 = 0 + \frac{1}{2} \times 4.33 \times 15^2 \\ &= 487.125 \text{ m}\end{aligned}$$

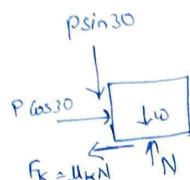
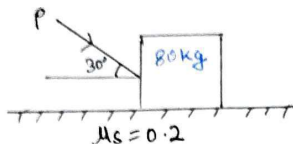
$$\begin{aligned}\textcircled{4} W &= F \cdot d = F \cos 30^\circ \cdot s \\ &= 50 \cos 30^\circ \times 487.125 \\ &= 21093.13 \text{ J}\end{aligned}$$

OR

$$\begin{aligned}W &= \Delta KE = \frac{1}{2} m v^2 - \frac{1}{2} m u^2 \\ &= \frac{1}{2} \times 10 \times 64.95^2 - 0 \\ &= 21093.13 \text{ J}\end{aligned}$$

Ques 2 A 80 kg block pushed by a force of 298 N, as given starting from the rest moves a distance of 10m in 3 sec. Compute.

- Coeff. of Kinetic friction
- Work done by friction.



(a)

$$\begin{aligned}\Sigma F_y = m a_y &= 0 \\ \Rightarrow P \sin 30 + W - N &= 0 \\ \Rightarrow N &= P \sin 30 + W \\ N &= 298 \sin 30 + 80 \times 9.81 = 933.8 \text{ N}\end{aligned}$$

$$F_s = \mu_s N = 0.2 \times 933.8 = 186.76$$

$$P \cos 30 = 298 \times \cos 30 = 258.07 \text{ N}$$

$\therefore P \cos 30 > F_k \Rightarrow$ Body is in motion.

$$\begin{aligned}\Sigma F_x &= m a_x \\ \Rightarrow F \cos 30 - F_k &= m a \\ 298 \cos 30 - \mu_k \cdot 933.8 &= 80 a \quad \text{--- (A)}\end{aligned}$$

$$\begin{aligned}\text{also } s &= ut + \frac{1}{2} a t^2 \\ \Rightarrow 10 &= 0 + \frac{1}{2} a 3^2 \\ \Rightarrow a &= 2.22 \text{ m/sec}^2 \quad \text{--- (B)}\end{aligned}$$

from (A) and (B)

$$\frac{298 \cos 30 - 80 \times 2.22}{933.8} = \mu_k$$

$$\mu_k = 0.086$$

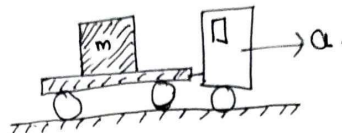
(b)

$$\begin{aligned}\text{work done } W &= \frac{1}{2} m v^2 - \frac{1}{2} m u^2 \quad \left| \quad F \cdot d \right. \\ &= F \cos 30 \times 10\end{aligned}$$

work done by friction

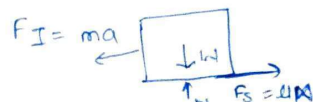
$$\begin{aligned}W_f &= -F_k \cdot s = \mu_k \cdot N \cdot s \\ &= -0.086 \times 933.8 \times 10 \\ &= -803.068 \text{ N}\end{aligned}$$

Ques 3:- A block is resting on rear of a truck initially at rest. If truck starts suddenly with an acceleration of 3 m/sec^2 , then find the required coefficient of friction b/w the block and truck to just prevent the sliding.



1st Approach - (D'Alembert) :-

$$\begin{aligned}\Sigma F_x &= 0 \\ \Rightarrow F - F_I &= 0 \\ \Rightarrow F &= F_I \\ \mu N &= m a \\ \mu &= \frac{m a}{N} \quad \text{--- (A)}\end{aligned}$$



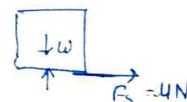
$$\begin{aligned}\Sigma F_y &= 0 \\ \Rightarrow W - N &= 0 \\ \Rightarrow W &= N \quad \text{--- (B)}\end{aligned}$$

from (A) and (B)

$$\mu = \frac{m a}{W} = \frac{m a}{m g} = \frac{a}{g} = \frac{3}{10} = 0.3 \quad \text{Ans}$$

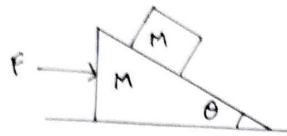
2nd Approach (Newton's Law)

$$\begin{aligned}\Sigma F_y &= m a_y = 0 \\ \Rightarrow W - N &= 0 \\ \Rightarrow N &= W\end{aligned}$$

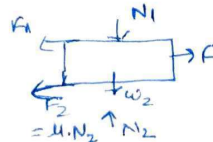
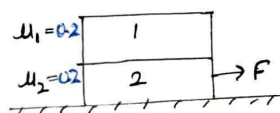
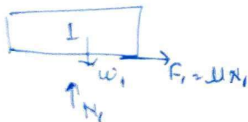


$$\begin{aligned}F_s &= \Sigma F_x = m a_x \\ F_s &= m a \\ \mu N &= m a \\ \mu W &= m a \\ \mu \cdot m g &= m a \\ \mu &= \frac{a}{g} = 0.3 \quad \text{Ans}\end{aligned}$$

Ques 4 Two blocks placed over each. After as shown, Find the required force 'F' to just prevent the sliding action b/w both blocks. Assume all the surfaces to be smooth.



Ques 5 Two blocks each of mass 0.5 kg are kept on a table over each other. The coefficient of friction for each pair of contact is 0.2. The lower block is pulled with a force "F". Compute its minimum value for which slip occurs b/w the two blocks.



$$\begin{aligned}\Sigma F_y &= m_1 a_y = 0 \\ \Rightarrow N_1 &= W_1 \\ \Sigma F_x &= m_1 a_x \\ \Rightarrow F_1 &= m_1 a_x \\ \Rightarrow \mu N_1 &= m_1 a_x \\ \Rightarrow a_x &= \frac{\mu N_1}{m_1}\end{aligned}$$

$$a_x = \frac{\mu \cdot W_1}{m_1}$$

$$a_x = \frac{\mu \cdot m_1 g}{m_1}$$

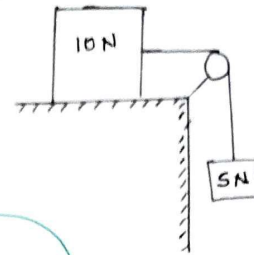
$$a_x = (0.2 \times 9.8)$$

$$a_x = 2 \text{ m/sec}^2$$

$$\begin{aligned}\Sigma F_y &= m_2 a_y = 0 \\ \Rightarrow N_1 + W_2 - N_2 &= 0 \\ N_2 &= N_1 + W_2 = W_1 + W_2 = 2mg\end{aligned}$$

$$\begin{aligned}\Sigma F_x &= m_2 a_x \Rightarrow F - F_1 - F_2 - m_2 a_x \\ F &= m_2 a_x + F_1 + F_2 \\ F &= m_2 a_x + \mu N_1 + \mu N_2 \\ F &= 0.5 \times 2 + 0.2 (0.5 \times 10 + 2 \times 0.5 \times 10) \\ F &= 1 + 3 = 4 \text{ N}\end{aligned}$$

Ques 6 Find the acceleration of 10 N block, when system is released from rest



$$\Sigma F_y = m_1 a_y = 0$$

$$\Rightarrow W_1 = N_1$$

$$\Rightarrow N_1 = (m_1 + m_2)g = 2mg \quad \text{--- (i)}$$

$$\Sigma F_x = m_1 a_x$$

$$F = (m_1 + m_2)a$$

$$F = 2m \cdot a \Rightarrow a = \frac{F}{2m} \quad \text{--- (ii)}$$

$$\Sigma F_{y'} = m_2 a_{y'}$$

$$N_2 = W_2 \cos \theta = m_2 a \sin \theta$$

$$N_2 = m_2 a \sin \theta + W_2 \cos \theta \quad \text{--- (iii)}$$

$$\Sigma F_{x'} = m_2 a_{x'}$$

$$W_2 \sin \theta = m_2 a \cos \theta$$

$$mg \frac{\sin \theta}{\cos \theta} = ma \Rightarrow a = g \tan \theta \quad \text{--- (iv)}$$

$$\text{from eq (ii) (iv)}$$

$$F = 2mg \tan \theta$$

$$\Sigma F_y = m_2 a_y$$

$$T - W_1 = m_1 (-a)$$

$$T - 5 = 0.5(-a)$$

$$T - 5 = -0.5a$$

$$5 - T = 0.5a \quad \text{--- (i)}$$

$$\Sigma F_y = m_2 a_y = 0$$

$$\Rightarrow N_2 - W_2 = 0$$

$$\Rightarrow N_2 = W_2$$

$$\Sigma F_x = m_2 a_x$$

$$T = m_2 a$$

$$T = \frac{10}{18} a$$

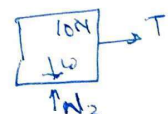
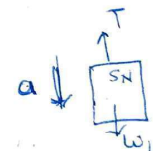
$$T = 1a \quad \text{--- (ii)}$$

$$\text{from (i) and (ii)}$$

$$5 - a = 0.5a$$

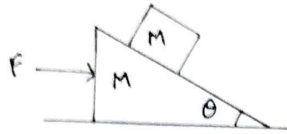
$$1.5a = 5$$

$$a = \frac{5}{1.5} = 3.33 \text{ m/sec}^2$$

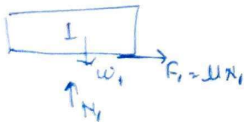


$$\begin{aligned}W &= mg \\ m &= \frac{W}{g}\end{aligned}$$

Ques 4 Two blocks placed over each. After as shown, Find the required force 'F' to just prevent the sliding action b/w both blocks. Assume all the surfaces to be smooth.



Ques 5 Two blocks each of mass 0.5 kg are kept on a table over each other. The coefficient of friction for each pair of contact is 0.2. The lower block is pulled with a force "F". Compute its minimum value for which slip occurs b/w the two blocks.



$$\sum F_y = m_1 a_y = 0$$

$$\Rightarrow N_1 = w_1$$

$$\sum F_x = m_1 a_x$$

$$\Rightarrow F_1 = m_1 a_x$$

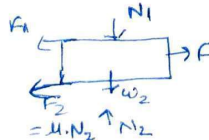
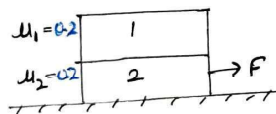
$$\Rightarrow \mu N_1 = m_1 a_x$$

$$a_x = \frac{\mu \cdot w_1}{m_1}$$

$$a_x = \frac{\mu \cdot m_1 g}{m_1}$$

$$a_x = (0.2 \times 9.8)$$

$$a_x \approx 2 \text{ m/sec}^2$$



$$\sum F_y = m_2 a_y = 0$$

$$\Rightarrow N_1 + w_2 - N_2 = 0$$

$$N_2 = N_1 + w_2 = w_1 + w_2 = 2mg$$

$$\sum F_x = m_2 a_x \Rightarrow F - F_1 - F_2 - m_2 a$$

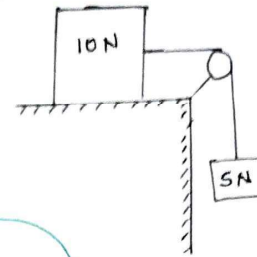
$$F = m_2 a + F_1 + F_2$$

$$F = m_2 a + \mu N_1 + \mu N_2$$

$$F = 0.5 \times 2 + 0.2 (0.5 \times 10 + 2 \times 0.5 \times 10)$$

$$F = 1 + 3 = 4 \text{ N}$$

Ques 6 Find the acceleration of 10 N block, when system is released from rest



$$\sum F_y = m_1 a_y = 0$$

$$\Rightarrow w_1 = N_1$$

$$\Rightarrow N_1 = (m_1 + m_2)g = 2mg \text{ --- (i)}$$

$$\sum F_x = m_1 a_x$$

$$F = (m_1 + m_2) a$$

$$F = 2m \cdot a \Rightarrow a = \frac{F}{2m} \text{ --- (ii)}$$

$$\sum F_y' = m a_y'$$

$$N_2 = w \cos \theta = m a \sin \theta$$

$$N_2 = m a \sin \theta + w \cos \theta \text{ --- (iii)}$$

$$\sum F_x' = m a_x$$

$$w \sin \theta = m a \cos \theta$$

$$m g \frac{\sin \theta}{\cos \theta} = m a \Rightarrow a = g \tan \theta \text{ --- (iv)}$$

$$\text{from eq (ii) (iv) } \boxed{F = 2m g \tan \theta}$$

$$\sum F_y = m a_y$$

$$T - w_1 = m_1 (-a)$$

$$T - 5 = 0.5(-a)$$

$$T - 5 = -0.5a$$

$$5 - T = 0.5a \text{ --- (i)}$$

$$\sum F_y = m_2 a_y = 0$$

$$\Rightarrow N_2 - w_2 = 0$$

$$\Rightarrow N_2 = w_2$$

$$\sum F_x = m_2 a_x$$

$$T = m_2 a$$

$$T = \frac{10}{10} a$$

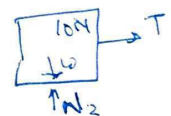
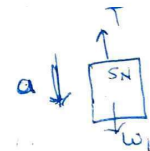
$$T = 1a \text{ --- (ii)}$$

from (i) and (ii)

$$5 - a = 0.5a$$

$$1.5a = 5$$

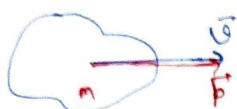
$$a = \frac{5}{1.5} = 3.33 \text{ m/sec}$$



LINEAR MOMENTUM CONSERVATION

LINEAR MOMENTUM ($\vec{p}$):-

It is a vector physical quantity given by the effect of mass and velocity of the system.



$$\begin{aligned}\vec{p} &= m \cdot \vec{v} \\ p_x &= m v_x \\ p_y &= m v_y \\ p_z &= m v_z\end{aligned}$$

LINEAR MOMENTUM CONSERVATION PRINCIPLE :-

If net external force on any body or system is zero, then its linear momentum would remain constant.

$$(\sum F_{\text{net}})_{\text{ext}} = ma$$

$$\text{If } (\sum F_{\text{net}})_{\text{ext}} = 0 \\ ma = 0$$

$$\text{If } m = \text{constant} \\ \Rightarrow a = 0 \Rightarrow v = \text{constant}$$

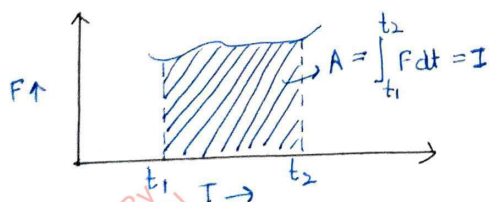
$$\text{Hence } \boxed{p = mv = \text{constant}}$$

IMPULSE and MOMENTUM:-

Impulse ($\vec{I}$):- When the force acts on a body for a small interval of time, it is called IMPULSE

$$\boxed{I = F \cdot \Delta t}$$

$$\text{OR } I = \lim_{\Delta t \rightarrow 0} F \cdot \Delta t = \int_{t_1}^{t_2} F \cdot dt \quad \text{N-sec.}$$



- Linear Impulse is also equal to the change in linear momentum.

$$F = ma = m \frac{dv}{dt} \quad \text{--- (i)}$$

$$\text{also, } I = \int_{t_1}^{t_2} F \cdot dt = \int_{t_1}^{t_2} m \frac{dv}{dt} dt = \int_u^v m dv = m(v-u)$$

$$I = mv - mu = \Delta p$$

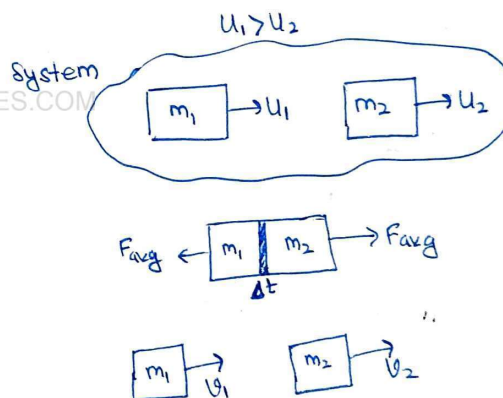
Hence -

$$\boxed{I = \int_{t_1}^{t_2} F dt = \Delta p = p_f - p_i}$$

IMPACT and COLLISION

When two elastic bodies, initially moving with some velocities strike to each other, firstly they deform then due to elasticity, will try to regain original shape and get separated with new velocities.

During the contact for short interval of time, a large force acts on these bodies termed as **IMPACT FORCE** and process is termed as **COLLISION**



Since, for the entire system $(\sum F_{\text{net}})_{\text{ext}} = 0$

$$\vec{p}_{\text{system}} = \vec{p}_{\text{system}}$$

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

Coefficient of Restitution (e)

- It is defined as, $e = - \frac{\text{Relative velocity after impact}}{\text{Relative velocity before impact}}$

$$e = - \frac{[v_2 - v_1]}{u_2 - u_1} = \frac{v_1 - v_2}{u_2 - u_1} = \frac{v_2 - v_1}{u_1 - u_2}$$

For Perfect elastic collision, $e = 1$
Imperfect elastic collision, $e = 0$

$$\text{Hence } \downarrow \\ 0 \leq e \leq 1$$

Average Impact Force ($\vec{F}_{avg}$)

$$I = F_{avg} \cdot \Delta t = \Delta p \Rightarrow F_{avg} = \frac{\Delta p}{\Delta t} = \frac{(p_f - p_i)_1}{\Delta t}$$

OR

$$\frac{(p_f - p_i)_2}{\Delta t}$$



SPECIAL CASES :-

(A) If $e=1$ and $m_1 = m_2 \Rightarrow$ Velocities are exchanged.

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

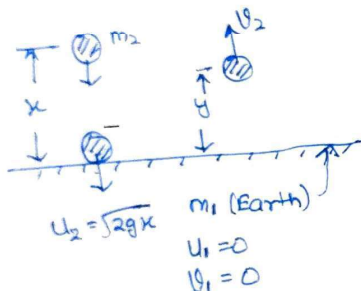
$$u_1 + u_2 = v_1 + v_2 \quad \text{--- (i)}$$

$$e = - \left[\frac{v_2 - v_1}{u_2 - u_1} \right] = 1 \Rightarrow u_2 - u_1 = v_1 - v_2 \quad \text{--- (ii)}$$

from (i) and (ii)

$$\left. \begin{array}{l} u_2 = v_1 \\ u_1 = v_2 \end{array} \right\} \text{velocity exchanged.}$$

(B) If $e=1$ and $m_1 \gg m_2 \Rightarrow$ Body rebounds with same velocity.



$$m_1 u_1 + m_2 (-u_2) = m_1 v_1 + m_2 v_2$$

$$-u_2 = v_2$$

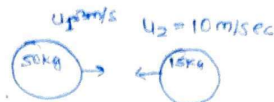
$$e = - \left[\frac{v_2 - v_1}{u_2 - u_1} \right] = 1$$

$$- \frac{(v_2 - 0)}{-u_2 - 0} = 1 \Rightarrow u_2 = v_2$$

also $x = y$ (From energy conservation)

UN-26

Ques (1) A body of 50 kg moving at 3 m/sec strikes to another body of mass 15 kg moving in just opposite direction, with 10 m/sec. If $e=0.5$, then find average impact force if it last for 0.1 sec.



$$F_{avg} \leftarrow \begin{array}{c} m_1 \quad m_2 \\ \Delta t = 0.1 \text{ sec} \end{array} \rightarrow F_{avg}$$



$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$50 \times 3 + 15 \times (-10) = 50 v_1 + 15 v_2$$

$$10 v_1 + 3 v_2 = 0 \quad \text{--- (i)}$$

$$e = 0.5 = - \frac{[v_2 - v_1]}{[u_2 - u_1]} = \frac{v_1 - v_2}{-10 - 3}$$

$$v_2 - v_1 = 6.5 \quad \text{--- (ii)}$$

from (i) and (ii)

$$v_1 = 1.5 \text{ m/s} \quad v_2 = -5 \text{ m/sec.}$$

$$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{(p_f - p_i)_1}{\Delta t} \quad \text{OR} \quad \frac{(p_f - p_i)_2}{\Delta t}$$

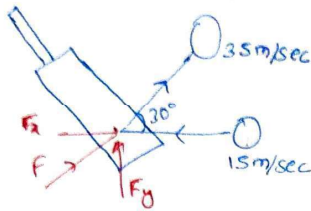
$$= \frac{m_1 v_1 - m_1 u_1}{\Delta t} \quad \text{OR} \quad \frac{m_2 v_2 - m_2 u_2}{\Delta t}$$

$$= \frac{50(3 - 1.5)}{0.1} \quad \text{OR} \quad \frac{15(-10 - (-5))}{0.1}$$

$$F_{avg} = 750 \text{ N} \quad \text{OR} \quad -750 \text{ N}$$

Written By
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Ques 2 A ball of 10 kg strikes to the bat with a velocity of 15 m/sec and leaves the bat with 35 m/sec making an angle 30° with the direction of initial velocity as shown. If contact happens for 0.1 sec. Then find the force applied by the bat on the ball.



$$F_{avg} = \frac{\Delta P}{\Delta t}$$

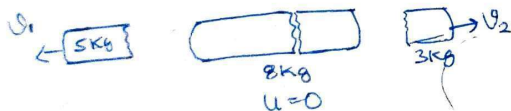
$$F_x = \frac{\Delta p_x}{\Delta t} = \frac{(p_f - p_i)_x}{\Delta t} = \frac{m(v_f - v_i)_x}{\Delta t}$$

$$= \frac{10 [35 \cos 30^\circ - (-15)]}{0.1} = 453 \text{ N}$$

$$F_y = \frac{\Delta p_y}{\Delta t} = \frac{(p_f - p_i)_y}{\Delta t} = \frac{m(v_f - v_i)_y}{\Delta t} = \frac{10 (35 \sin 30^\circ - 0)}{0.1} = 1750 \text{ N}$$

$$F = \sqrt{F_x^2 + F_y^2} = 1857.2 \text{ N}$$

Ques 3 A body of mass "m" initially at rest, is suddenly burst into two parts due to residual stress. Mass of two parts are 5 kg and 3 kg. respectively. Assuming that the parts fly in just opposite direction with relative speed of 25 m/sec. Find the speed of each part



$$mu = m_1 u_1 + m_2 u_2$$

$$0 = 5u_1 + 3(-u_2)$$

$$5u_1 = 3u_2 \quad \text{--- (i)}$$

Relative speed $u_2 = 25 \text{ m/sec}$

$$u_1 - (-u_2) = 25 \text{ m/sec}$$

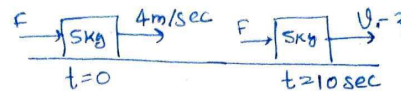
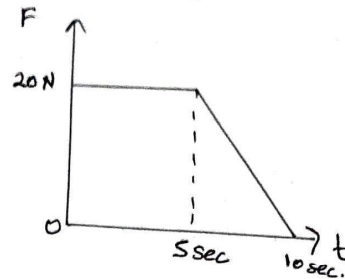
$$u_1 + u_2 = 25 \text{ m/sec} \quad \text{--- (ii)}$$

from (i) and (ii)

$$u_1 = 9.375 \text{ m/sec}$$

$$u_2 = 15.625 \text{ m/sec}$$

Ques 4 A block of mass 5 kg, initially moving with a velocity of 4 m/sec is subjected to a variable force as given. Find the velocity of block after 10 sec



$$\text{Impulse (I)} = \int_{t_1}^{t_2} F \cdot dt = \text{Area under F-t diagram}$$

$$\text{So, } I = \Delta P \text{ (change in momentum)}$$

$$\Rightarrow \text{Area under curve} = \text{change in momentum}$$

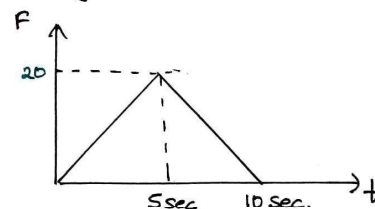
$$(20 \times 5) + \frac{1}{2} \times 5 \times 20 = m \cdot u_1 - m \cdot u_i$$

$$100 + 50 = 5(u_1 - 4)$$

$$\frac{150}{5} = u_1 - 4$$

$$u_1 = \frac{150}{5} + 4 = 34 \text{ m/sec. Ans}$$

Ques 5 Variation of force on a block of mass 10 kg is given, what is the final velocity of the block if initially it was at rest.



from above

$$\text{Area under curve (F-t)} = \text{change in momentum}$$

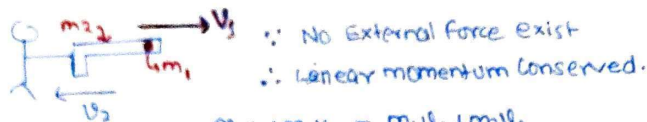
$$\frac{1}{2} (10) \times 20 = 10(u - 0)$$

$$u = 10 \text{ m/sec. Ans}$$

UN-27

Ques 6:— A bullet of 100gm is fired from a gun of 5kg with a velocity of 150m/sec, then find—

- Impulse applied by gun on man
- Force applied by man on the gun to stop it in 0.2sec.



$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$(0.1 \times 0) + (5 \times 0) = (0.1 \times 150) + (5 \times v_2)$$

$$0 = 15 - 5v_2$$

$$v_2 = 3 \text{ m/sec.}$$

(a)

Now:— Impulse = $\int_{t_1}^{t_2} F dt = \Delta p = (p_f - p_i)_{\text{gun}} = m_2 v_2 - m_2 u_2$

$$= 5 \times 3 = 15 \text{ N-sec}$$

(b)

$$\int_0^{0.2} F dt = 15 \text{ N-sec}$$

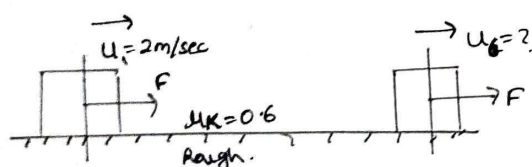
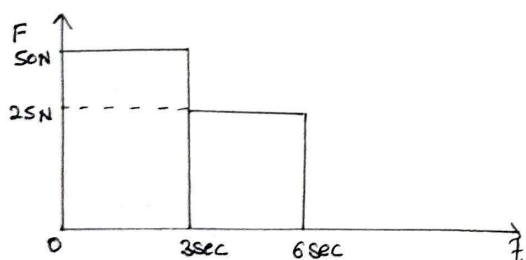
$$\Rightarrow F(0.2) = 15$$

$$F = 75 \text{ N.}$$



Very space

Ques 7 A 10kg block is moving to the right with 2m/sec on plane. If a force 'F' starts acting at $t=0$, then find the velocity of block after 6sec, Variation of the force is given the graph.



$$\sum F_y = ma_y = 0 \Rightarrow N = W = 100 \text{ N} \quad (i)$$

$$I = \int_{t_1}^{t_2} F dt = \text{Area of } F-t = \Delta p = p_f - p_i$$

$$\int_0^6 (F - F_k) dt = \Delta p$$

$$\int_0^6 F dt - \int_0^6 F_k dt = \Delta p$$

$$50 \times 3 + 25 \times 3 - 20 \times 6 = 10(v_6 - 2)$$

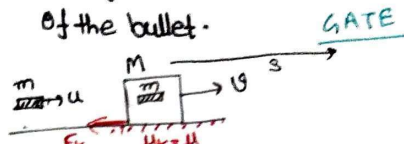
$$v_6 = 12.5 \text{ m/sec}$$

also $F_k = \mu_k N = \mu_k W$

$$= 0.6 \times 100 \text{ N}$$

$$= 60 \text{ N.}$$

Ques 8 A block of mass 'm' is initially at rest on rough surface ($\mu_k = \mu$) a bullet of mass 'm' moving with 'u' hit the block and gets embedded into it and whole system moves on the surface up to distance 's', then find the Initial velocity 'u' of the bullet.



$$mu + M \times 0 = (m+M) v$$

$$v = \frac{mu}{m+M} \quad (i)$$

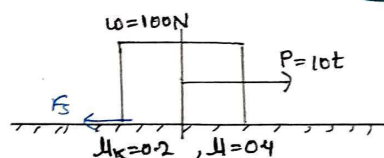
also $W = \Delta KE$

$$-F_k s = \frac{1}{2} (m+M) v^2 - \frac{1}{2} (m+M) 0^2$$

$$-\mu (m+M) g s = -\frac{1}{2} (m+M) \left(\frac{mu}{m+M} \right)^2$$

$$u = \frac{m+M}{m} \sqrt{2gs}$$

Ques 9 A block of 100N is initially at rest on rough surface. If a variable force $P = 10t$ is applied on the block, then find its velocity after 15 sec.



$$\sum F_y = ma_y = 0 \Rightarrow N = W = 100 \text{ N}$$

$$\sum F_x = 0 \Rightarrow P - F_s = 0 \Rightarrow P = F_s$$

$$\Rightarrow 10t = 0.4 \times 100 \Rightarrow t = 4 \text{ sec.}$$

Hence, from $t=0$ to $t=4 \text{ sec}$, block is at rest.

Now $\int_{t_1}^{t_2} F dt = \Delta p$

$$\Rightarrow \int_4^{15} (P - F_k) dt = p_f - p_i$$

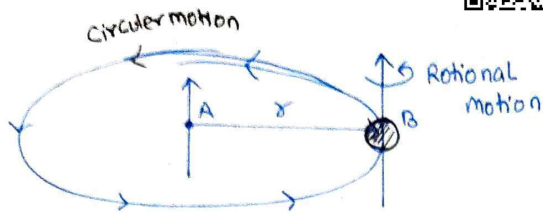
$$\Rightarrow \int_4^{15} (10t - 0.2 \times 100) dt = m(v_{15} - v_4)$$

$$\left| \frac{10t^2}{2} - 20t \right|_4^{15} = \frac{100}{10} v_{15}$$

$$v_{15} = \left(\frac{15^2 - 4^2}{2} \right) - 2(15 - 4)$$

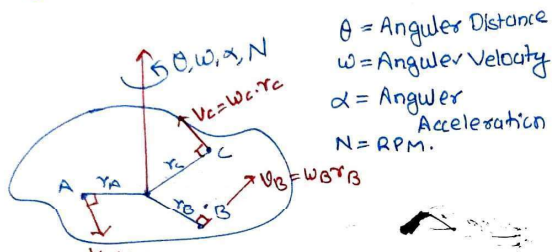
$$v_{15} = 32.5 \text{ m/sec.}$$

ROTATIONAL MOTION



When any rigid body moves/revolves about an axis passing through its own C.G. is called rotational motion.

In case of circular motion size/dimension of body are neglected, but in case of Rotational motion the size of body is considered.



Angular Quantity
 $\theta_A = \theta_B = \theta_C = \theta$
 $\omega_A = \omega_B = \omega_C = \omega$
 $\alpha_A = \alpha_B = \alpha_C = \alpha$
 $N_A = N_B = N_C = N$

Linear Quantity
 $v_A \neq v_B \neq v_C$
 $a_A \neq a_B \neq a_C$

ω = initial angular velocity
 ω_0 = final angular velocity

Type of Rotation :-

a) $\alpha = 0$, $\omega = \text{const} \Rightarrow \omega = \frac{\theta}{t}$ Rad/sec.

b) $\alpha = \text{const}$, $\omega \neq \text{const} \Rightarrow \omega = \omega_0 + \alpha t$
 $\omega^2 = \omega_0^2 + 2\alpha\theta$
 $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

c) $\alpha \neq \text{const}$, $\omega \neq \text{const}$ for equation
 Example 2
 $\theta = 4t^4$

$$\omega = \frac{d\theta}{dt} = 16t^3$$

$$\alpha = \frac{d\omega}{dt} = 48t^2$$

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$$\omega = \frac{2\pi N}{60}$$

Ques (1) A disc of radius 1m, starts a rotating from rest about its center and achieve a speed of 40 rpm in 25 sec uniformly then compute:-

- Angular acceleration
- Linear velocity of point 'P' on periphery after 10 sec.
- Total linear acceleration of point 'P' after 15 sec.
- No. of revolution made in 1000 sec.

(a) $\omega_0 = 0$, $\alpha = \text{const. (uniformly)}$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 40}{60} = 4.18 \text{ Rad/sec.}$$

$$\omega = \omega_0 + \alpha t$$

$$4.18 = 0 + \alpha (25)$$

$$\alpha = 0.1672 \text{ rad/sec}^2.$$

(b) $\omega = \omega_0 + \alpha t = 0 + 0.1672 \times 10 = 1.672 \text{ rad/sec}$

$$v_{10\text{sec}} = r\omega_{10\text{sec}} = 1 \times 1.672 = 1.672 \text{ m/sec}$$

(c) $\omega_{15\text{sec}} = \omega_0 + \alpha t = 0 + 0.1672 \times 15 = 2.508 \text{ rad/sec}$

Tangential acceleration
 $a_{t, 15\text{sec}} = \alpha_{15\text{sec}} \cdot r = 0.1672 \times 1 = 0.1672 \text{ m/sec}^2$

Tangential acceleration = angular acceleration $\times r$

Tangential velocity = angular velocity $\times r$

Centrifugal Acceleration = (angular acceleration) $^2 \times r$

$$a_c = \omega^2 \cdot r = (2.508)^2 \times 1 = 6.29 \text{ m/sec}^2.$$

centrifugal acceleration

Total Linear acceleration = $\sqrt{(\text{tangential acceleration})^2 + (\text{centrifugal acceleration})^2}$

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{(0.1672)^2 + (6.29)^2}$$

$$a = 6.92 \text{ m/sec.}$$

(d) $\theta_{1000\text{sec}} = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(0.1672)(1000)^2$

$$\theta_{1000\text{sec}} = 83600 \text{ Radian.}$$

$$n_{1000} = \frac{83600}{2\pi} = 13305 \text{ Ans}$$

Ques 1 - A flywheel rotating at 3 rev/sec its speed increases at a constant rate of 50 rev/min each second, during an interval of 10 sec, No. of revolution it has made during this time would be how much.

$$\omega_0 = 3 \times 2\pi = 6\pi \text{ Rad/sec}$$

$$\alpha = \frac{50 \times 2\pi}{60} = 1.66\pi \text{ Rad/sec}$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = (6\pi \times 10) + \frac{1}{2} \times 1.66\pi (10)^2$$

$$\theta = 10 \times 14.3\pi$$

$$n = \frac{\theta}{2\pi} = \frac{10 \times 14.3\pi}{2\pi} = 71.5 \text{ Revolution}$$



UN-23

Ques 2 Angle of rotation of a body is given by equation of motion as

$$\theta = \omega t^3 - 10t^2 + 16t + 32.$$



Compute -

a) Angular velocity at time $t=0 \text{ sec}, t=4 \text{ sec}$

b) Angular acceleration at time $t=0 \text{ sec}, t=4 \text{ sec}.$

$$a) \omega = \frac{d\theta}{dt} = 12t^2 - 20t + 16$$

$$\omega_{at=0} = 16 \text{ Rad/sec}$$

$$\omega_{at=4} = 12(4)^2 - 20(4) + 16 = 128 \text{ Rad/sec}$$

$$b) \alpha = \frac{d\omega}{dt} = 24t - 20$$

$$\alpha_{at=0} = -20 \text{ Rad/sec}^2$$

$$\alpha_{at=4 \text{ sec}} = 24(4) - 20 = 78 \text{ Rad/sec}^2$$

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Ques 3 - A Flywheel of 460 mm dia is brought uniformly to a speed of 280 rpm in 15 sec from rest. Find the velocity and acceleration of point "A" on the circumference after '1 sec'.

$$\omega_{15} = \frac{2\pi N t}{60} = \frac{2 \cdot \pi \cdot 280}{60} = 9.33\pi \text{ Rad/sec}$$

$$\omega_t = \omega_0 + \alpha t = 0 + \alpha(15)$$

$$\alpha = \frac{9.33\pi}{15} = 0.622\pi \text{ Rad/sec}$$

$$a) v = \omega r = (\omega_0 + \alpha t)r = (0 + 0.622\pi \times 1)0.23$$

$$v = 0.45 \text{ m/sec}$$

$$b) a_t = \alpha \cdot r = 0.622\pi \times 0.23 = 0.449 \text{ m/sec}^2$$

$$a_c = r \cdot \omega^2 = 0.23 \times (0.622\pi)^2 = 0.878 \text{ m/sec}^2$$

$$a = \sqrt{a_t^2 + a_c^2}$$

$$a = \sqrt{0.449^2 + 0.878^2} = 0.988 \text{ m/sec}^2$$

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A. HANNAN

Kinetics of Rotational Motion! -



(A) Torque ($\vec{\tau}$) :-

It is a vector quantity given by the rate of change of angular momentum of the body about any point/axis.

$$\vec{\tau} = \frac{d(\vec{Q})}{dt} = \frac{d(I \cdot \vec{\omega})}{dt} = I \frac{d\vec{\omega}}{dt} + \vec{\omega} \frac{dI}{dt} \quad \text{--- (i)}$$

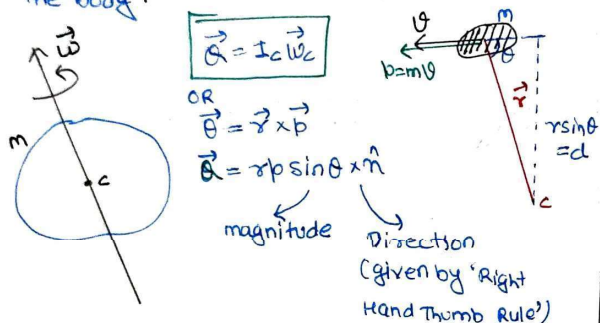
Basic Equation.

If $m = \text{constant} \Rightarrow I = \text{constant} \Rightarrow \vec{\tau} = I \frac{d\vec{\omega}}{dt} = I \vec{\alpha}$ --- (A)

If $\omega = \text{constant}$, $\vec{\tau} = \vec{\omega} \frac{dI}{dt}$ --- (B)

(B) Angular momentum ($\vec{Q}$)

It is a vector quantity given by the effect of m and angular velocity of the body.



$$Q = r \sin \theta p = p d.$$

It is also given by 'moment of linear momentum' of any body about point or axis.

If $\theta = 90^\circ$, $Q = pr = mvr$

(C) Work Energy Principle! - change in kinetic energy.

As per this principle $W = \Delta KE$

$$W = \frac{1}{2} I (\omega_2^2 - \omega_1^2)$$

Ques 1 A ring flywheel of $m = 100 \text{ kg}$, $r = 10 \text{ m}$, initially at rest is applied with a tangential force $F = 1000 \text{ N}$ for 10 sec . Compute -

- Angular acceleration
- Angular velocity
- Angular displacement.
- Work done in 10 sec
- Apply work energy principle for 10 sec .

(a) $T = F \times r = 1000 \times 10 = 10^4 \text{ N-m}$ --- (i)

$T = I \alpha = m r^2 \alpha$ --- (ii)

from (i) and (ii)

$100 (10)^2 \alpha = 10^4$
 $\alpha = 10 = 10 \text{ Rad/sec}^2$

(b) $\omega = \omega_0 + \alpha t = 0 + 10 \times 10 = 10 \text{ Rad/sec}$

(c) $\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = 0 + \frac{1}{2} \times 10 \times 10^2 = 500 \text{ rad}$

(d) $W = T \cdot \theta = 10^4 + 50 \times 10^3 = 500 \text{ kJ}$

(e) $W = \Delta KE$
 $10^3 \times 500 = \frac{1}{2} I (\omega_2^2 - 0)$
 $\omega_2 = 10 \text{ Rad/sec}$

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Ques ② A Flywheel of mass 5000kg having radius of gyration of 1m, decreases its speed from 400 rpm to 200 rpm in 4 min uniformly. Compute-

- Retarding Torque
- Change in Kinetic energy
- Change in Angular momentum

$$\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 400}{60} = 41.88 \text{ Rad/sec}$$

$$\omega_2 = \frac{2\pi N_2}{60} = \frac{2\pi \times 200}{60} = 20.94 \text{ Rad/sec.}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$20.94 = 41.88 + \alpha (4 \times 60)$$

$$\alpha = 0.087 \text{ Rad/sec}$$

Retardation m.

$$\textcircled{a} T = I\alpha = mr^2\alpha = 5000(1)^2 \times 0.087 = 435 \text{ N-m}$$

$$\textcircled{b} \Delta KE = \frac{1}{2} I (\omega_2^2 - \omega_1^2)$$

$$= \frac{1}{2} \times 5000 (20.94^2 - 41.88^2)$$

$$= 32.88 \text{ kJ}$$

$$\textcircled{c} \Delta L = L_2 - L_1 = I\omega_2 - I\omega_1$$

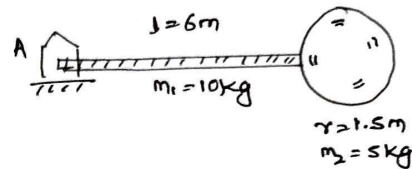
$$= I(\omega_2 - \omega_1)$$

$$= 5000 \times 1^2 (20.94 - 41.88)$$

$$= 104700 \text{ Kg-m}^2/\text{Sec.}$$

Unit-30 - Example 31

Ques ③ 1- A sphere is of radius 1.5m welded with a rod of length $l = 6m$, if the assembly is released from rest. Then find its instantaneous initial angular acceleration about the hinge. mass of Rod $m_1 = 10 \text{ kg}$, mass of sphere $m_2 = 5 \text{ kg}$



Torque about A

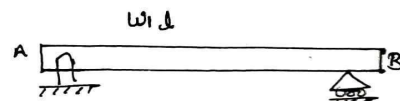
$$T_A = I_A \alpha = \omega_1 r_1 + \omega_2 r_2 \rightarrow (6 + 1.5) m$$

$$\left[\left(\frac{m_1 l^2}{3} \right) + \left(\frac{2}{5} m_2 l^2 + m_2 r^2 \right) \right] \alpha = m_1 g r_1 + m_2 g r_2$$

mmol Rod about A mmol Sphere about A

$$\alpha = 0.67 \text{ Rad/sec}^2$$

Ques ④ A bar of weight 'W' and length 'l' is initially at rest in horizontal position with the help of supports at A and B. If support 'B' is suddenly removed, then find its initial instantaneous angular acceleration about hinge A. also compute instantaneous vertical reaction at A.



Sol:

For Rest Condition

$$\sum F_x = 0 \Rightarrow R_{Ax} = 0$$

$$\sum F_y = 0 \Rightarrow R_{Ay} + R_{By} = W$$

$$\sum M_A = 0 \Rightarrow R_{By} \cdot l = W \cdot \frac{l}{2} \Rightarrow R_{By} = \frac{W}{2} \text{ and } R_{Ay} = \frac{W}{2}$$

For dynamic Condition

$$\sum F_x = 0 \Rightarrow R_{Ax} - F_{Ix} = 0$$

$$R_{Ax} = F_{Ix} = m a_c = 0$$

$$T_A = I_A \alpha$$

$$W \cdot \frac{l}{2} = \frac{m l^2}{3} \alpha \Rightarrow \alpha = \frac{3g}{2l} = 1.5 \frac{g}{l} \text{ Rad/sec}^2$$

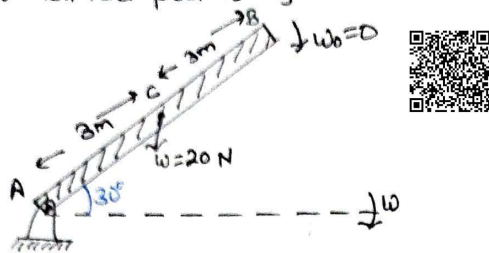
$$\text{also } \sum F_y = m a_y \Rightarrow R_{Ay} - W = m(a_y) = -m \gamma \alpha$$

$$R_{Ay} = W - m \cdot \frac{1.5g}{2} = W - 0.75W$$

$$R_{Ay} = 0.25W \text{ Ans}$$

Written By
A.HANNAN

Ques 5 Find the Vertical and horizontal axis at hinge "A" when bar reaches in horizontal position. It is released from initial inclined position from rest.



$$W = \Delta KE \Rightarrow W \times 3 \sin 30 = \frac{1}{2} I (\omega^2 - \omega_0^2)$$

$$20 \times 3 \times \frac{1}{2} = \frac{1}{2} \times \frac{20}{10 \times 3} \times 6^2 \omega^2$$

$$\Rightarrow \omega = 1.581 \text{ Rad/sec.}$$

$$T = I_A \cdot \alpha$$

$$W \times 3 = \frac{m(6)^2}{3} \alpha$$

$$\frac{10 \times 3 \times 3}{6^2} = \alpha$$

$$\Rightarrow \alpha = 2.5 \text{ Rad/sec}^2$$

$$\Sigma F_x = m a_x \Rightarrow R_{Ax} = m(-a_c) = -m r \omega^2$$

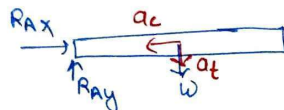
$$= -\frac{20}{10} \times 3 \times 1.58^2$$

$$R_{Ax} = -15 \text{ N}$$

$$\Sigma F_y = m a_y \Rightarrow R_{Ay} - W = m(-a_t) \Rightarrow R_{Ay} = W - m(r\alpha)$$

$$R_{Ay} = 20 - \frac{20}{10} \times 3 \times 2.5$$

$$R_{Ay} = 5 \text{ N.}$$



Unit-32

SIMPLE HARMONIC MOTION (SHM)

Periodic motion:-/ Harmonic motion:-

If a body or system repeats its motion after fixed interval of time, then its motion is termed as Periodic OR Harmonic motion.



Oscillatory motion:-

If a body or system repeats its motion on some path again and again it is termed as Oscillatory motion. (watch)

Simple Harmonic motion:-

It is a special type of oscillatory motion in which body moves under the influence of restoring force such that -

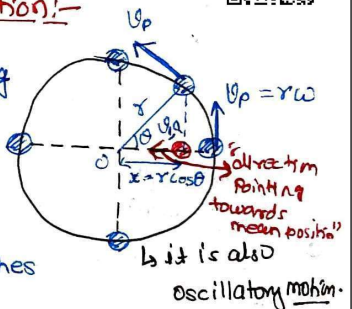
- Its acceleration is directly proportional to the displacement from mean position.
- Its acceleration is always directed towards the mean position.



Simple Harmonic motion:-

- Here, particle is moving in a circular motion with uniform velocity

$$\omega = \frac{\theta}{t} \Rightarrow \theta = \omega t$$



- Now, Displacement of the particle

$$x = r \cos \theta$$

- velocity $v = \frac{dx}{dt} = -r \sin \theta \frac{d\theta}{dt}$

$$v = -r \omega \sin \theta$$

- acceleration $a = \frac{dv}{dt} = -r \omega \cos \theta \frac{d\theta}{dt}$

$$a = -r \omega^2 \cos \theta$$

Now $a = -\omega^2 x$ $\because x = r \cos \theta$

Hence $a \propto x$ and $a = (-ve)$

↓
proportional to displacement

↓
Towards the mean position.

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→ Hence when particle performs uniform circular motion, its projection on horizontal-axis perform simple Harmonic motion.

Equation of sum: $a = -x\omega^2$

$$\frac{d^2x}{dt^2} = -x\omega^2$$

For linear motion: $\frac{d^2x}{dt^2} + x\omega^2 = 0$

Similarly:-

For Angular motion: $\frac{d^2\theta}{dt^2} + \theta\omega^2 = 0$

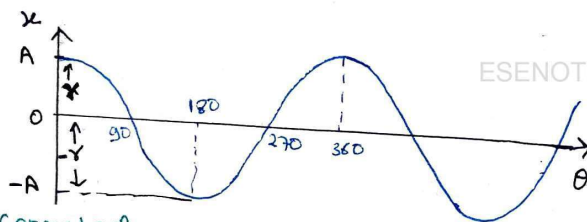
- circular frequency = ω (rad/sec)

- Time period for complete revolution/cycle: $T = \frac{2\pi}{\omega}$ (Sec.)

- Frequency (No. of cycles/sec)
 $f = \frac{1}{T} = \frac{\omega}{2\pi}$ (Hz)

ANALYSIS :-

[A] Displacement:-



General eqⁿ of displacement $x = A \cos(\omega t + \phi)$ ϕ = phase angle.

- at $t=0, \theta=0 \Rightarrow x = x_{\max} = A = r$ (Amplitude)
- at $t = \frac{T}{4}, \theta = 90^\circ, x = x_{\min} = 0$
- at $t = \frac{T}{2}, \theta = 180^\circ, x = x_{\max} = -r = -A$ (Amplitude)

[B] Velocity ($v = -r\omega \sin\theta$)

at $t=0, \theta=0; v = v_{\min} = 0$ Direction

at $t = \frac{T}{4}, \theta = 90^\circ; v = v_{\max} = -r\omega$ magnitude.

at $t = \frac{T}{2}, \theta = 180^\circ; v = v_{\min} = 0$

at $t = \frac{3T}{4}, \theta = 270^\circ; v = v_{\max} = r\omega$

$$v = v_{\max} \sin(\omega t + \phi)$$

$\phi \rightarrow$ phase angle.

[C] Acceleration ($a = -r\omega^2 \cos\theta$)

at $t=0, \theta=0; a = a_{\max} = -r\omega^2$

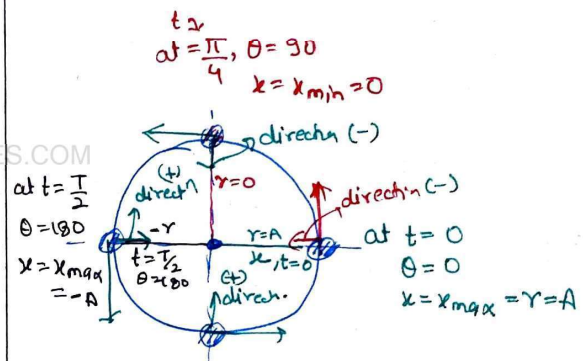
at $t = \frac{T}{4}, \theta = 90^\circ; a = a_{\min} = 0$

at $t = \frac{T}{2}, \theta = 180^\circ; a = a_{\max} = r\omega^2$

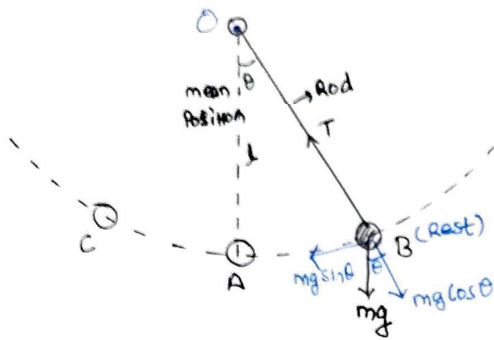
at $t = \frac{3T}{4}, \theta = 270^\circ; a = a_{\min} = 0$

$$a = a_{\max} \cos(\omega t + \phi)$$

$\phi \rightarrow$ Phase Angle.



SIMPLE - PENDULUM



For every small angle " θ ", simple pendulum also performs 'simple harmonic motion'

$$T_0 = I_0 \alpha_0$$

$$mg \sin \theta \cdot l = ml^2 \alpha$$

$$g \sin \theta = l \alpha$$

$$g \sin \theta = l \left(-\frac{d^2 \theta}{dt^2} \right)$$

If θ is small $\Rightarrow \theta \approx \sin \theta \approx \tan \theta$

$$\Rightarrow g \theta = l \left(-\frac{d^2 \theta}{dt^2} \right)$$

$$\Rightarrow \frac{l d^2 \theta}{dt^2} + g \theta = 0 \rightarrow \text{Equation of SHM}$$

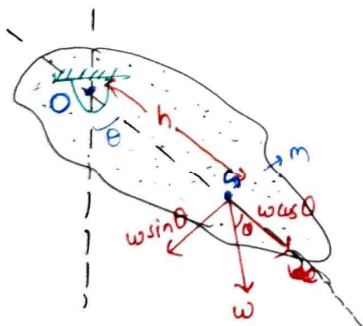
$$\Rightarrow \frac{d^2 \theta}{dt^2} + \frac{g \theta}{l} = 0 \xrightarrow{\text{Relate actual eqn of SHM}} \frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0$$

$$\therefore \omega^2 = \frac{g}{l} \Rightarrow \omega = \sqrt{\frac{g}{l}}$$

$$(\text{Time Period}) T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}$$

$$(\text{Frequency}) f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g}{l}}$$

Compound Pendulum! ->



When a complete rigid body is hanged at some point and it can oscillate about that point, it is termed as Compound pendulum.

$$T_0 = I_0 \alpha_0$$

$$mg \sin \theta \cdot h = [I_0 + mh^2] \alpha$$

$$mg \sin \theta \cdot h = [mk^2 + mh^2] \alpha$$

If θ is small, $\theta \approx \sin \theta \approx \tan \theta$

$$g \theta h = [k^2 + h^2] \alpha$$

$$g \theta h = (k^2 + h^2) \left(-\frac{d^2 \theta}{dt^2} \right)$$

"K = Radius of gyration"

$$(k^2 + h^2) \frac{d^2 \theta}{dt^2} + g \theta h = 0$$

$$\frac{d^2 \theta}{dt^2} + \frac{gh}{(k^2 + h^2)} \theta = 0 \xrightarrow{\text{Relate actual eqn of SHM}} \frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0$$

$$\omega^2 = \frac{gh}{k^2 + h^2}$$

$$\omega = \sqrt{\frac{g}{\frac{k^2 + h^2}{h}}} = \sqrt{\frac{g}{L_e}}$$

$$\left[\text{Where } L_e = \frac{k^2 + h^2}{h} \right]$$

$L_e \rightarrow$ Equivalent length.

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L_e}{g}}$$

For max. and min. Time period,

$$\frac{dL_e}{dh} = 0$$

$$\frac{d}{dh} \left(\frac{k^2 + h^2}{h} \right) = 0$$

$$\Rightarrow \frac{d}{dh} \left(\frac{k^2}{h} + h \right) = 0$$

$$\Rightarrow -\frac{k^2}{h^2} + 1 = 0$$

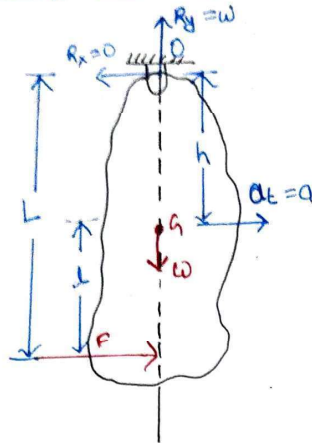
$$\Rightarrow h^2 = k^2$$

$$\Rightarrow h = k$$

$$T_{\min} = 2\pi \sqrt{\frac{2k}{g}}$$

K = Radius of gyration

Centre of Percussion (COP)



It is the point where if we apply a horizontal force on hinged body, then there will be no horizontal reaction on the hinge.

$$T_0 = I_0 \alpha$$

$$FL = (I_G + mh^2) \alpha$$

$$FL = (mk^2 + mh^2) \alpha \quad \text{--- (i)}$$

also $\alpha_t = a = r\alpha = h\alpha \quad \text{--- (ii)}$

from (i) and (ii)

$$FL = m(k^2 + h^2) \frac{a}{h} \quad \text{--- (iii)}$$

$$\sum F_x = ma_x$$

$$F - R_x = ma_x = ma$$

$$\Rightarrow a = \frac{F}{m} \quad \text{--- (iv)}$$

from (ii) and (iv)

$$FL = m(k^2 + h^2) \frac{F}{m} \cdot \frac{1}{h}$$

$$L = \frac{k^2 + h^2}{h}$$

$$\text{also } L_e = \frac{k^2 + h^2}{h}$$

Hence, Distance of Centre of Percussion (COP) from hinge is same as equivalent length of compound pendulum

$$L = L_e$$

↳ equivalent length of compound pendulum

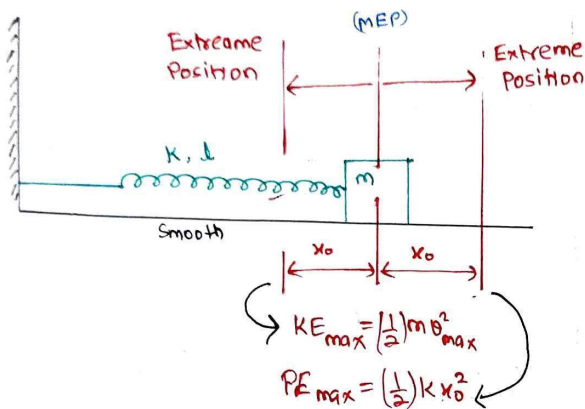
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VIBRATION:-



A body is said to be in vibration if it performs "to and fro" motion about a mean equilibrium position. (MEP)

Vibration occurs due to restoring elastic forces where in the energy of the system continuously changes from KE to PE



One Oscillation/One Cycle:-

When the body reaches at its same initial position during motion, it is termed as cycle.

Time Period:-

The time taken by the body or system to complete one cycle or one oscillation is termed as time period.

Frequency:-

It is the number of cycles complete in 1-second.

Resonance:-

When system vibrates under the influence of some periodic external force, then if the frequency of applied force become equal to the natural frequency of the system, the amplitude of vibration become very large due to which system may fail and process is termed as Resonance.

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Types of Vibration:-

a) Free/Natural Vibration:-

When there is no external force like friction, air drag after providing the initial disturbance to the system and system vibrates freely, it is termed as free/natural vibration.

b) Damped Vibration:-

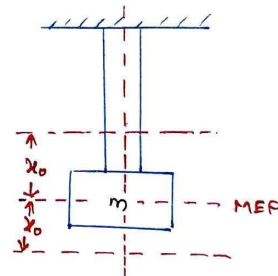
When the energy of system continuously dissipates through the resistive force like friction, air drag, etc, the system slowly comes to rest at initial mean position after some time and it is termed as Damped vibration.

c) Forced Vibration:-

When a repeated external force, continuously acts on system by some external agent, it is termed as Forced Vibration.

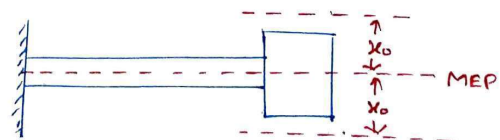
d) Longitudinal Vibration:-

When the particle of system moves along the longitudinal axis, it is termed as Longitudinal Vibration.



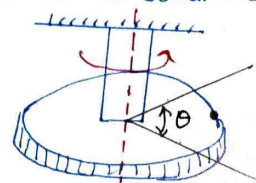
e) Lateral Vibration:-

When the particle of the system moves perpendicular to the longitudinal axis of system it is termed as Lateral/Transverse Vibration



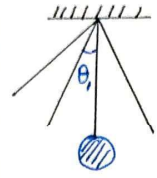
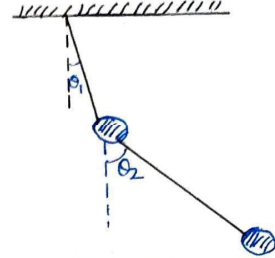
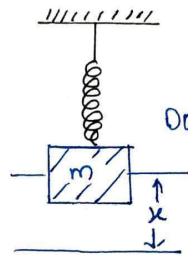
f) Torsional vibration:-

When the particles on a circular path about the longitudinal axis, it is termed as Torsional Vibration.

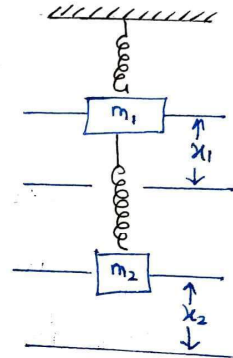
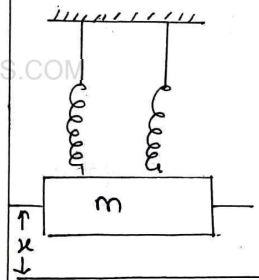


DEGREE OF FREEDOM

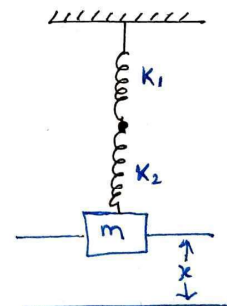
The number of independent variables or co-ordinates required to define the position of system at any instant is termed as Degree of Freedom of the system.

DOF = 1 (θ_1)DOF = 2 (θ_1, θ_2)

DOF: 1(x)

DOF: 2 (x_1, x_2)

DOF: 1 (m)



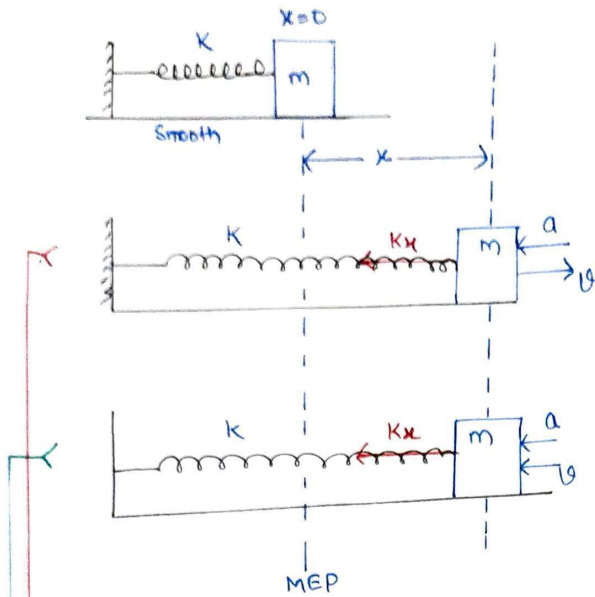
DOF: 1 (x)

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FREE & NATURAL VIBRATION

↳ No resistive force



$$v = \frac{dx}{dt}, \quad a = -\frac{dv}{dt} = -\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\frac{d^2x}{dt^2}$$

$$v = -\frac{du}{dt}, \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(-\frac{dx}{dt}\right) = -\frac{d^2x}{dt^2}$$

Since the acceleration is always towards MEP, it is SHM

$$\sum F_x = ma_x$$

$$-Kx = -ma_x$$

$$Kx = m\left(-\frac{d^2x}{dt^2}\right) = -m\ddot{x}$$

$$\left(\ddot{x} = \frac{d^2x}{dt^2}\right)$$

$$m\ddot{x} + Kx = 0 \quad \text{compare with actual eqn of SHM}$$

$$\ddot{x} + \frac{Kx}{m} = 0 \Leftrightarrow \ddot{x} + \omega^2 x = 0$$

$$\Rightarrow \omega^2 = \omega_n^2 = \frac{K}{m} \Rightarrow \omega_n = \sqrt{\frac{K}{m}}$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{K}}; \quad f_n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\text{also } \sum F_x = ma_x$$

$$-Kx = m\left(-v\frac{dv}{dx}\right)$$

$$Kx dx = -m \int v dv$$

$$\frac{Kx^2}{2} = -\frac{mv^2}{2} + C$$

$$\frac{1}{2}Kx^2 + \frac{mv^2}{2} = C$$

{ Conservation of Energy }

now at extreme position,

$$PE \rightarrow \max, \quad KE = 0$$

at ME position, $PE = 0, \quad KE \rightarrow \max$.

$$\text{Hence } \frac{1}{2}Kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}mv_{\max}^2 \quad (\text{at MEP})$$

$$v = \sqrt{v_{\max}^2 - \frac{K}{m}x^2}$$

$$\frac{dx}{dt} = \sqrt{v_{\max}^2 - \frac{K}{m}x^2}$$

MEP
↳ mean
equilibrium
position

$$\int \frac{dx}{\sqrt{v_{\max}^2 - \frac{K}{m}x^2}} = \int dt \quad \left[\int \frac{1}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} \sin^{-1} \frac{bx}{a} \right]$$

$$\frac{1}{\sqrt{\frac{K}{m}}} \sin^{-1} \left[\sqrt{\frac{K}{m}} \frac{x}{v_{\max}} \right] = t + C$$

$$\sin^{-1} \left[\sqrt{\frac{K}{m}} \frac{x}{v_{\max}} \right] = \sqrt{\frac{K}{m}} t + \phi$$

$$\sqrt{\frac{K}{m}} \frac{x}{v_{\max}} = \sin \left(\sqrt{\frac{K}{m}} t + \phi \right)$$

$$\frac{\omega_n x}{v_{\max}} = \sin(\omega_n t + \phi)$$

$$x = \frac{v_{\max}}{\omega_n} \sin(\omega_n t + \phi)$$

This is
equation of
free SHM
(general eqn)

$$x = A \sin(\omega t + \phi)$$

A = Amplitude
 ϕ = phase
angle

General equation of SHM

If motion starts from MEP, $\phi = 0, t = 0, x = 0$

$$x = A \sin \omega t$$

If motion starts from extreme position

$$t = 0, x = A, \phi = \frac{\pi}{2}$$

$$x = A \left(\sin \omega t + \frac{\pi}{2} \right)$$

$$x = A \cos \omega t$$

P.T.O

- Let the motion start from extreme position

$$x = A \cos \omega t$$

$$v = \frac{dx}{dt} = -A \sin \omega t (\omega) = -A\omega \sin \omega t$$

$$v = A\omega \cos \left(\omega t + \frac{\pi}{2} \right)$$

Hence phase difference b/w 'x' and 'v' is $\frac{\pi}{2}$

$$a = \frac{dv}{dt} = -A\omega \cos \omega t (\omega) = -A\omega^2 \cos \omega t$$

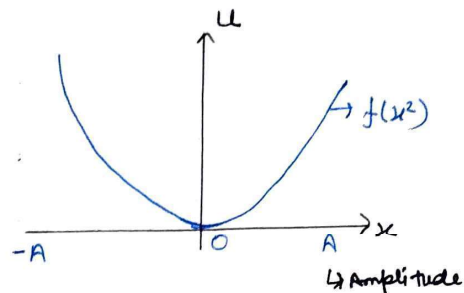
$$a = A\omega^2 \cos \left(\omega t + \pi \right)$$

Hence phase difference b/w 'x' and 'a' is π
and phase difference b/w 'v' and 'a' is $\frac{\pi}{2}$

Potential Energy in SHM

- For any spring mass system PE is saved in the spring in the form strain energy

$$U = \frac{1}{2} Kx^2 = f(x^2)$$



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- Let the motion start from extreme position

$$x = A \cos \omega t$$

$$v = \frac{dx}{dt} = -A \sin \omega t (\omega) = -A\omega \sin \omega t$$

$$v = A\omega \cos \left(\omega t + \frac{\pi}{2} \right)$$

Hence phase difference b/w 'x' and 'v' is $\frac{\pi}{2}$

$$a = \frac{dv}{dt} = -A\omega \cos \omega t (\omega) = -A\omega^2 \cos \omega t$$

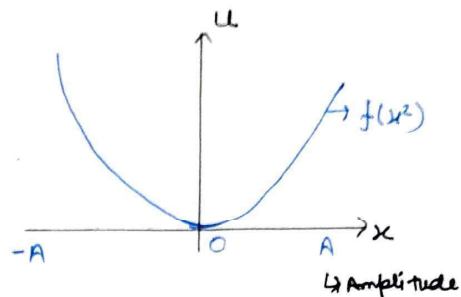
$$a = A\omega^2 \cos \left(\omega t + \pi \right)$$

Hence phase difference b/w 'x' and 'a' is π
and phase difference b/w 'v' and 'a' is $\pi/2$

Potential Energy in SHM

- For any spring mass system PE is saved in the spring in the form strain energy

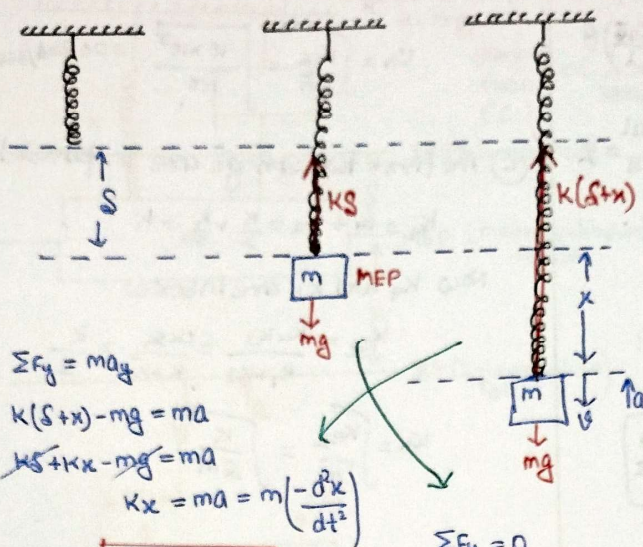
$$U = \frac{1}{2} Kx^2 = f(x^2)$$



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Spring-mass-system In Vertical Position



$$\Sigma F_y = ma_y$$

$$K(S+x) - mg = ma$$

$$KS + Kx - mg = ma$$

$$Kx = ma = m \left(\frac{d^2x}{dt^2} \right)$$

$$\ddot{x} + \frac{K}{m}x = 0 \quad \text{SHM}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{g}{S}}$$

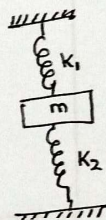
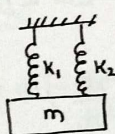
$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{K}} = 2\pi \sqrt{\frac{S}{g}}$$

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g}{S}}$$

- If the springs are in parallel.

$$K_{eq} = K_1 + K_2$$

$$\omega_n = \sqrt{\frac{K_{eq}}{m}}$$

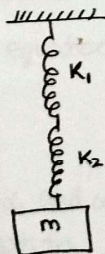


- If springs are in series

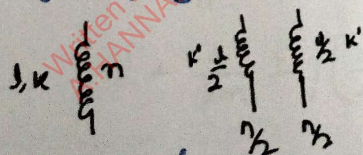
$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

$$\omega_n = \sqrt{\frac{K_{eq}}{m}}$$



- If the spring is cut down into two half

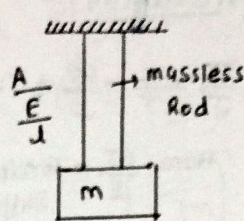


$$K = 2K'$$

$$K \propto \frac{1}{n}$$

n = number of turns in spring

Rod-mass System:-



$$S = \frac{(mg)l}{AE}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{g}{S}} = \sqrt{\frac{gAE}{mg l}} = \sqrt{\frac{AE}{ml}}$$

Here $\frac{AE}{l}$ is called axial stiffness of rod/bar

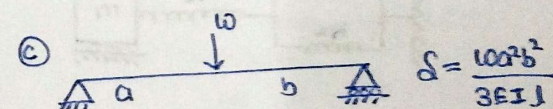
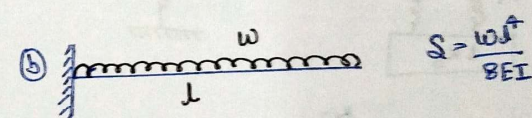
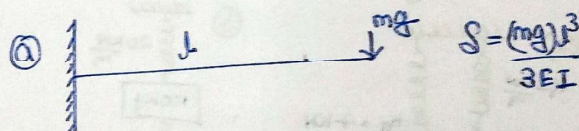
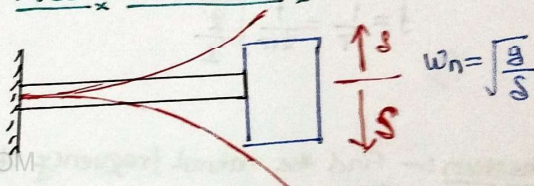
$$\text{Let } \frac{AE}{l} = S \Rightarrow \omega_n = \sqrt{\frac{S}{m}}$$

→ If mass of rod is also considered then take equivalent mass (M_e)

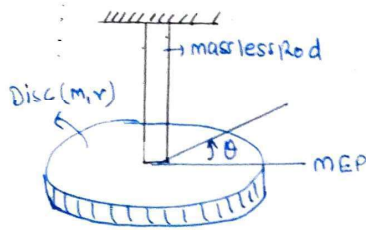
$$M_e = m + \frac{m'}{3}$$

m' = mass of Rod

Free Transverse Vibration:-



Free Torsional Vibration:-



$$\frac{T}{J} = \frac{Z}{r} = \frac{G\theta}{L} \Rightarrow T = \left(\frac{GJ}{L}\right)\theta$$

Here $\frac{GJ}{L}$ = Torsional Stiffness = 2

$$T = 2\theta \quad \text{--- (i)}$$

$$\text{also } T = I\alpha \quad \text{--- (ii)}$$

from (i) and (ii)

$$2\theta = I\alpha = I \left(\frac{d^2\theta}{dt^2} \right)$$

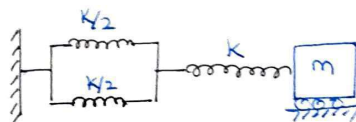
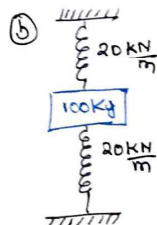
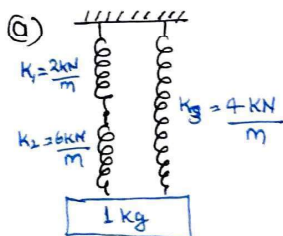
$$\ddot{\theta} + \frac{2\theta}{I} = 0 \quad \text{SHM} \quad \ddot{\theta} = \frac{d^2\theta}{dt^2}$$

$$\omega_n = \sqrt{\frac{2}{I}}$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{I}{2}}$$

$$I = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{2}{I}}$$

Question:- Find the natural frequency of the system.



(a) As K_1 and K_2 are in series; $K_4 = \frac{K_1 K_2}{K_1 + K_2} = \frac{2 \times 6}{2 + 6} = \frac{12}{8}$

$$K_4 = \frac{3}{2} \text{ kN/m.}$$

As K_4 and K_3 are in parallel; $K_5 = K_4 + K_3$

$$= \frac{12}{8} + 4 = \frac{11}{2} \text{ kN/m.}$$

$$\omega_n = \sqrt{\frac{K_5}{m}} = \sqrt{\frac{\frac{11}{2} \times 10^3}{1}} = 74.16 \text{ Rad/sec.}$$

(b) As spring are in parallel

$$K_{eq} = 20 + 20 = 40 \text{ kN/m}$$



$$\omega_n = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{40 \times 10^3}{100}} = 20 \text{ Rad/sec.}$$

(c) As first two springs are in parallel

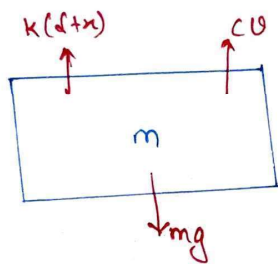
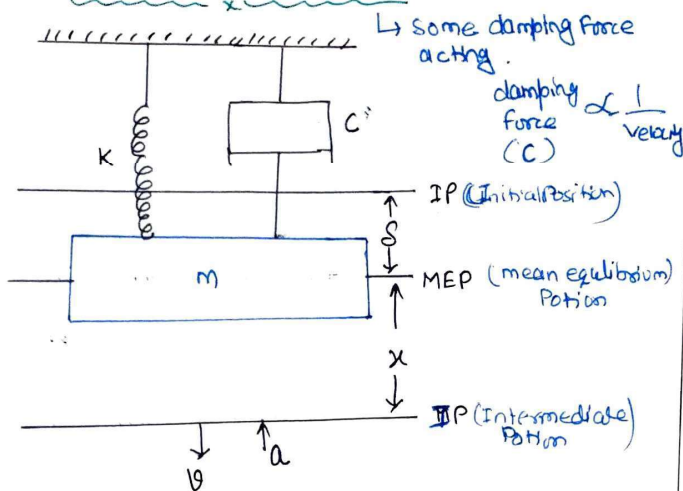
$$K_4 = K_1 + K_2 = \frac{K}{2} + \frac{K}{2} = K$$

Now K_4 and K_3 are in series

$$K_{eq} = \frac{K_4 K_3}{K_4 + K_3} = \frac{K \cdot K}{K + K} = \frac{K}{2}$$

$$\omega_n = \sqrt{\frac{K_{eq}}{m}} = \sqrt{\frac{K}{2m}}$$

DAMPED VIBRATION



At equilibrium position $\Sigma F_y = 0$
 $mg = Ks$ — (i)

At Intermediate position
 $\Sigma F_y = ma_y$

$$K(s+x) + Cv - mg = ma$$

$$\cancel{Ks} + Kx + Cv - \cancel{mg} = ma$$

$$Kx + Cv = ma$$

$$Kx + C \left(\frac{dx}{dt} \right) = m \left(-\frac{d^2x}{dt^2} \right)$$

$$m\ddot{x} + C\dot{x} + Kx = 0$$

$$\ddot{x} + \frac{C}{m}\dot{x} + \frac{K}{m}x = 0 \quad \text{Equation of SHM}$$

Solution of this differential equation is given by —

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

A, B are constants and α_1 and α_2 are the roots of following equation.

$$\alpha^2 + \frac{C}{m}\alpha + \frac{K}{m} = 0$$

$$\alpha = \frac{-C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \left(\frac{K}{m}\right)}$$

Now —

(a) If $\left(\frac{C}{2m}\right)^2 > \frac{K}{m} \Rightarrow \text{Over Damping}$

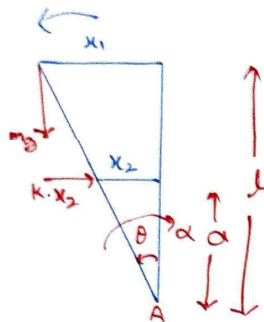
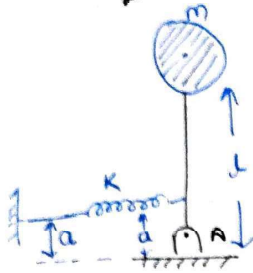
(b) If $\left(\frac{C}{2m}\right)^2 < \frac{K}{m} \Rightarrow \text{Under Damping}$

(c) If $\left(\frac{C}{2m}\right)^2 = \frac{K}{m} \Rightarrow \text{Critical Damping}$



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Ques 1 A massless rod attached with point mass "m" and a spring of stiffness "k" is given. Find the frequency of natural vibration and also compute the condition for stable equilibrium.



From similar Δ

$$\frac{x_2}{a} = \frac{x_1}{l} \quad \text{--- i}$$

$$x_1 = l\theta$$

$$x_2 = \frac{x_1 a}{l} = \frac{l\theta a}{l} = \theta a$$

$$T_A = I_A \alpha$$

$$mgx_1 - Kx_2 a = ml^2(-\alpha)$$

$$mgl\theta - K\theta a \cdot a = -ml^2\alpha$$

$$(mgl - Ka^2)\theta = -ml^2\alpha$$

$$ml^2\alpha + (mgl - Ka^2)\theta = 0$$

$$ml^2\ddot{\theta} + (mgl - Ka^2)\theta = 0$$

$$\theta + \left(\frac{mgl - Ka^2}{ml^2} \right) \theta = 0$$

Eqn of SHM

Natural Frequency

$$\omega_n^2 = \frac{mgl - Ka^2}{ml^2}$$

$$\omega_n = \sqrt{\frac{mgl - Ka^2}{ml^2}}$$

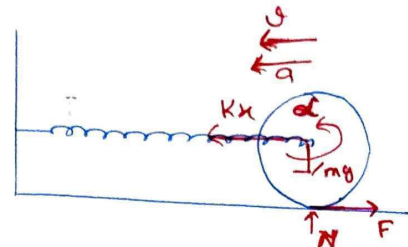
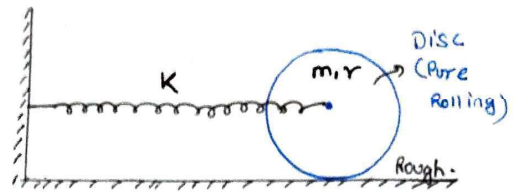
Condition for stable equilibrium :-

$$mgl - Ka^2 > 0$$

$$K < \frac{mgl}{a^2}$$

Written By
A. HANNAN

Ques 2 Find the circular natural frequency of the system.



$$\Sigma F_x = ma_x$$

$$-Kx + F = m(-a)$$

$$Kx - F = ma \quad \text{--- i}$$

$$\text{For pure rolling } a = r\alpha \quad \text{--- ii}$$

$$T_o = I_o \alpha \Rightarrow Fr = \frac{mr^2}{2} \alpha$$

$$\Rightarrow F = \frac{mr\alpha}{2} = \frac{ma}{2} \quad \text{--- iii}$$

From i and ii

$$Kx - \frac{ma}{2} = ma$$

$$Kx = \frac{3ma}{2}$$

$$K(r\theta) = \frac{3m}{2} r \cdot \alpha$$

$$K\theta = \frac{3m}{2} \left(-\frac{d^2\theta}{dt^2} \right) \Rightarrow \frac{3m}{2} \frac{d^2\theta}{dt^2} + K\theta = 0$$

$$\ddot{\theta} + \frac{2K}{3m} \theta = 0 \quad \text{Eqn of SHM}$$

$$\omega_n = \sqrt{\frac{2K}{3m}}$$

For Pure Rolling

$$x = r\theta$$

$$\theta = r\omega$$

$$a = r\alpha$$